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A STRATEGIC ANTISUBMARINE OFFENSIVE MINING MODEL (U)

BY

A. M. R. JOYE

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A.M.R. JOYE

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ANNEXES

1. INTRODUCTION

This note describes in detail mathematical models used in Ref. 1 to assess the effectiveness of an ASW offensive mining campaign. The models developed here apply to strategic mining (as opposed to tactical mining) since the effects of mining are averaged over the length of the mining campaign. A fundamental assumption is that submarines which are damaged or sunk cannot be repaired or replaced; the time taken to do so therefore dictates the maximum length of the mining campaign over which the models are valid.

An important feature of the models presented here is that they can be applied to very short mining campaigns (and hence very short wars) during which submarines only go on one or two patrols. For such short mining campaigns it is found that results from these models are significantly different to those from models based on the assumption of long term mining operations. As the length of the mining campaign increases however, the difference becomes negligible and the "long term" result, which is usually mathematically simpler, can be used.

For short mining campaigns, the submarine position (in port, in transit, or on patrol) at the onset of mining has a critical effect on the final result. For this reason, the result from each model is expressed in terms of an expected value, together with lower and upper bounds.

The strategic situation examined here is one where enemy submarines are sent regularly from their base to a patrol area; on transit, they encounter at least one minefield. Minefields can be positioned at the submarine base and/or at choke points such as straits. A typical such situation is depicted in Fig. 1.1. When faced with a minefield, enemy submarines have three options:

a. they can go through the minefield and suffer the risk of being damaged or sunk,

b. they can go around the minefield (if this is possible) and suffer increased transit times,

c. they can wait until mine countermeasures (MCM) forces have cleared the minefield, and thereby suffer a delay.

The above options form the basis for the three models developed in this note.

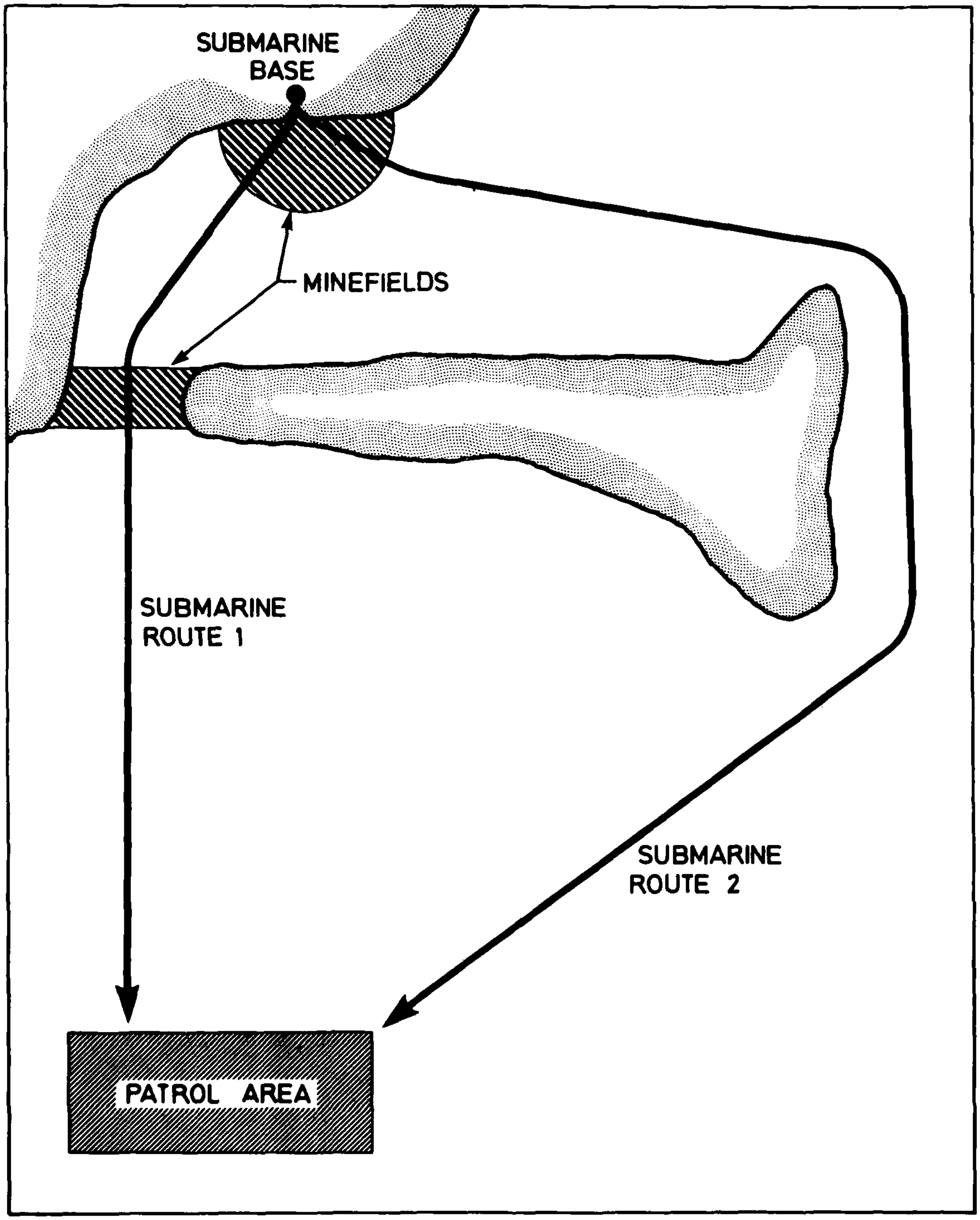

Fig. 1·1. Typical ASW offensive mining situation.

The three mining models are described in Section 3. In each case, mathematical formulae are given for the expected value $\overline{\Delta S}$ and for the lower and upper bounds ΔS_ℓ and ΔS_u ($\overline{\Delta S}$, ΔS_ℓ, and ΔS_u are discussed in Section 2). Simplifying approximations are presented and their applicability investigated. Long term model solutions are also presented and their validity is discussed. Derivations for each model are given in Annexes B, C and D.

The employment of the above models is discussed in the light of a number of possible mining options. Each mining option is examined (Section 4) and a decision theory approach for determining the optimum mining policy is described (Section 5).

2. MEASURE OF EFFECTIVENESS

The aim of the strategic ASW mining campaign is taken to be the reduction in the average number of submarines which the enemy can maintain on station in the patrol area. The success of such a mining campaign can therefore be assessed by using the fractional reduction in the number of submarines on station as a measure of effectiveness. However, a better assessment can be made by adding some refinements.

If enemy submarines follow a policy of going around the minefield, or wait for the minefield to be cleared by MCM forces, then the delays introduced will reduce the total number of days that can be spent on station. This effect is felt immediately. On the other hand, if the submarines follow a policy of going through the minefield then there is the possibility of a submarine casualty and the number of operational submarines will be reduced. Clearly, the reduction in submarine numbers (and hence the reduction in submarines on station) will increase as a function of time, as more and more minefield crossings are made. In order to make a comparison between this submarine policy and the former two possible, it is necessary to average the submarine reduction over the length of the mining campaign.

It is found that the initial submarine position (in port, in transit or on patrol), when the mining campaign begins, has an important effect on the result obtained. (This effect is particularly marked for short mining campaigns). For example, if a submarine has just left its base before it is mined, it can complete a full patrol before encountering the minefield. On the other hand, if the submarine base is mined just

before a submarine returns from patrol, the submarine will have to negotiate the minefield twice before its next patrol. In order to cover all such eventualities, it is necessary to sum the results for each submarine position and average over a complete operational cycle (base, transit, station, transit, base, etc). The resulting expected value represents the assumption that the submarines are uniformly randomly distributed within an operational cycle.

Given the above considerations, the measure of effectiveness adopted in this note is the expected average fractional reduction in the number of submarines on station, and is denoted by $\overline{\Delta S}$.

In practice, it is unlikely that enemy submarines will be randomly distributed. For example, if warned of an impending mining campaign, the submarines may be prepositioned in the patrol area before mining begins. Conversely, intelligence information on submarine movements may make it possible to mine when the majority of submarines are in port. To take such possibilities into account, lower and upper bounds, ΔS_{ℓ} and ΔS_{u}, are also determined. A more complete assessment of the effectiveness of the mining campaign is thus possible.

3. MATHEMATICAL MODELS

Three mathematical models are required to assess the three submarine options:

a. to traverse the minefield,

b. to go around the minefield,

c. to wait until the minefield is cleared.

In this section, the basis of each model is described and the solutions are expressed in terms of $\overline{\Delta S}$, ΔS_{ℓ}, and ΔS_{u}. In each case a number of simplifying approximations are given and these are compared with the "exact" solutions. A long-term mining model is also developed for each case, and its applicability is examined. Derivations for each model are given in Annexes B, C and D.

3.1 COMMON CONCEPTS

All models described in this note are based on a submarine operational cycle of length T_{CP}, described in Annex A. The length of the mining campaign is denoted by T_L and, to simplify model derivation,

is divided into two parts. The first part consists of the maximum number, N, of complete operational cycles that can occur in time T_L; the second part is the remaining number of days, T_R, which make up the last, uncompleted, operational cycle. Thus,

$$T_R = T_L - NT_{CP} \tag{3.1}$$

where
$$N = \mathrm{Int}\left[\frac{T_L}{T_{CP}}\right] \tag{3.2}$$

The number of times that a submarine is on patrol during time T_L is denoted by n. If no mining takes place, the expected number of times on patrol is

$$n' = N + \frac{T_R}{T_{CP}} = \frac{T_L}{T_{CP}} \tag{3.3}$$

When mining does take place, the expected number of times on patrol is $E(n)$. The expected average fractional reduction in the number of submarines on station is defined by

$$\overline{\Delta S} = 1 - \frac{E(n)}{n'} \tag{3.4}$$

Similarly the lower and upper bounds are defined by

$$\Delta S_\ell = 1 - \frac{n_u}{n'} \tag{3.5}$$

and
$$\Delta S_u = 1 - \frac{n_\ell}{n'} \tag{3.6}$$

Clearly, to obtain solutions for $\overline{\Delta S}$, ΔS_ℓ, and ΔS_u, it is necessary to determine solutions for $E(n)$, n_ℓ and n_u for each model.

3.2 MINEFIELD IS TRAVERSED

The case where submarines decide to go through the minefield, accepting the risk of a submarine casualty, is analysed in terms of a probabilistic model. It is assumed that the probability, p, of a submarine casualty per crossing is constant. The probability of a submarine being sunk (or damaged) on its i^{th} crossing follows the geometric

distribution,

$$Pr(i) \quad = \quad pq^{i-1} \tag{3.7}$$

where $q = 1 - p$.

The probability of a submarine completing exactly n patrols, P(n patrols), is equivalent to the probability of being sunk between the n^{th} patrol and the $n^{th} + 1$ patrol. Since there are two minefield crossings between consecutive patrols,

$$P(n\ \text{patrols}) \quad = \quad Pr(i) + Pr(i+1) \tag{3.8}$$

where the value of i is related to n as indicated in Tables B.1 to B.4.

For the special case of a mining campaign lasting for exactly N patrols (i.e. $T_L = NT_{CP}$ and $T_R = 0$) the expected number of submarine patrols is given by

$$E(n) \quad = \quad N\ P(N\ \text{patrols}) + \sum_{n=1}^{N-1} n\ P(n\ \text{patrols}) \tag{3.9}$$

where $$P(N\ \text{patrols}) \quad = \quad 1 - \sum_{n=0}^{N-1} P(n\ \text{patrols}) \quad . \tag{3.10}$$

Note that the probability of exactly N patrols, P(N patrols), is equivalent to the probability of the submarine surviving until the N^{th} patrol. An equation similar to eqn. 3.9 can be used to determine E(n) for any value of T_L (see Annex B).

The solution for E(n), eqn. B.27, is derived in Annex B. When the result for E(n) is substituted into eqn. 3.4, the solution for $\overline{\Delta S}$ is

$$\begin{aligned}
\overline{\Delta S} \quad = \quad & 1 - \frac{1}{T_L}\left[T_{CP}\, q + \frac{1}{2}(T_P + 2T_{T2})(1-q)^2\right] \cdot \left(\frac{1-q^{2N}}{1-q^2}\right) \\
& - \frac{q^{2N}}{T_P T_L}\left\{G(T_{T2}, T_{T2}, T_P) + G(0, T_P, T_P) - \frac{1}{2}T_P^{\,2}\right. \\
& + q \cdot G(T_{CP} - T_P - T_{T2}\,, T_{CP} - T_P - 2T_{T2}, T_P) \\
& \left. + q^2\left[G(T_{CP}, T_P, T_P) + G(T_{CP} - T_P, T_{T2}, T_P)\right]\right\}
\end{aligned} \tag{3.11}$$

where the function G(y,z,m) is defined by eqn. F.3 in Annex F, and parameters T_{CP}, T_P, and T_{T2} are defined in Annex A. Lower and upper bounds are given by (Annex B),

$$\Delta S_{\ell} = 1 - \frac{1}{N}\left[\frac{1-q^{2N}}{1-q^2}\right] \tag{3.12}$$

$$\Delta S_u = 1 - \frac{q^2}{N}\left[\frac{1-q^{2N}}{1-q^2}\right] \tag{3.13}$$

for the special case $T_L = NT_{CP}$ The increased complexity of expressions for ΔS_{ℓ} and ΔS_u for all values of T_L is not considered justified. However, values of ΔS_{ℓ} and ΔS_u for intermediate values of T_L can be obtained by interpolation (see also Subsection 3.2.1).

Figure 3.1 shows results for $\overline{\Delta S}$ as a function of T_L; results are also shown for ΔS_{ℓ} and ΔS_u for $N = 1, 2,$ etc. All curves in Fig. 3.1 are for a minefield threat level $p = 0.2$. Curve A represents a typical submarine operational cycle of length 50 days, consisting of 5 days in port $(T_B = 5)$, 25 days in the patrol area $(T_P = 25)$, and a total of 20 days in transit $(T_T = 10)$, with the minefield half-way between the submarine base and the patrol area $(T_{T1} = T_{T2} = 5)$. It can be seen that, as expected, $\overline{\Delta S}$ increases as T_L increases. Curves B and C represent extreme values for $\overline{\Delta S}$, which are obtained when the time spent in the patrol area is extremely short $(T_P \to 0)$. For curve B, the minefield is located near the approaches to the patrol area $(T_{T2} = 0)$ and the submarines spend most of their time either in port or on transit (i.e. $(T_B + 2T_{T1}) \to T_{CP})$. Note that since the probability of a submarine being in the patrol area, when the mining campaign begins, is very small, $\overline{\Delta S} \simeq p$ when $T_L \le T_{CP}$ since the minefield will almost certainly have to be crossed once before reaching the patrol area. For curve C, the minefield is located at the submarine base and the submarines spend most of their time in transit (i.e. $2T_T = 2T_{T2} \to T_{CP}$, and $T_B \to 0$). Note that since most of the submarines on their way to the patrol area will have already passed the minefield area when the mining campaign begins, $\overline{\Delta S} = 0$ when $T_L \le T_{T2}$. Although curves B and C represent highly unrealistic situations, these curves show the maximum range of values that $\overline{\Delta S}$ can acquire. It can be seen that although this range is large for $T_L < T_{CP}$, it rapidly diminishes as T_L increases; thus, for

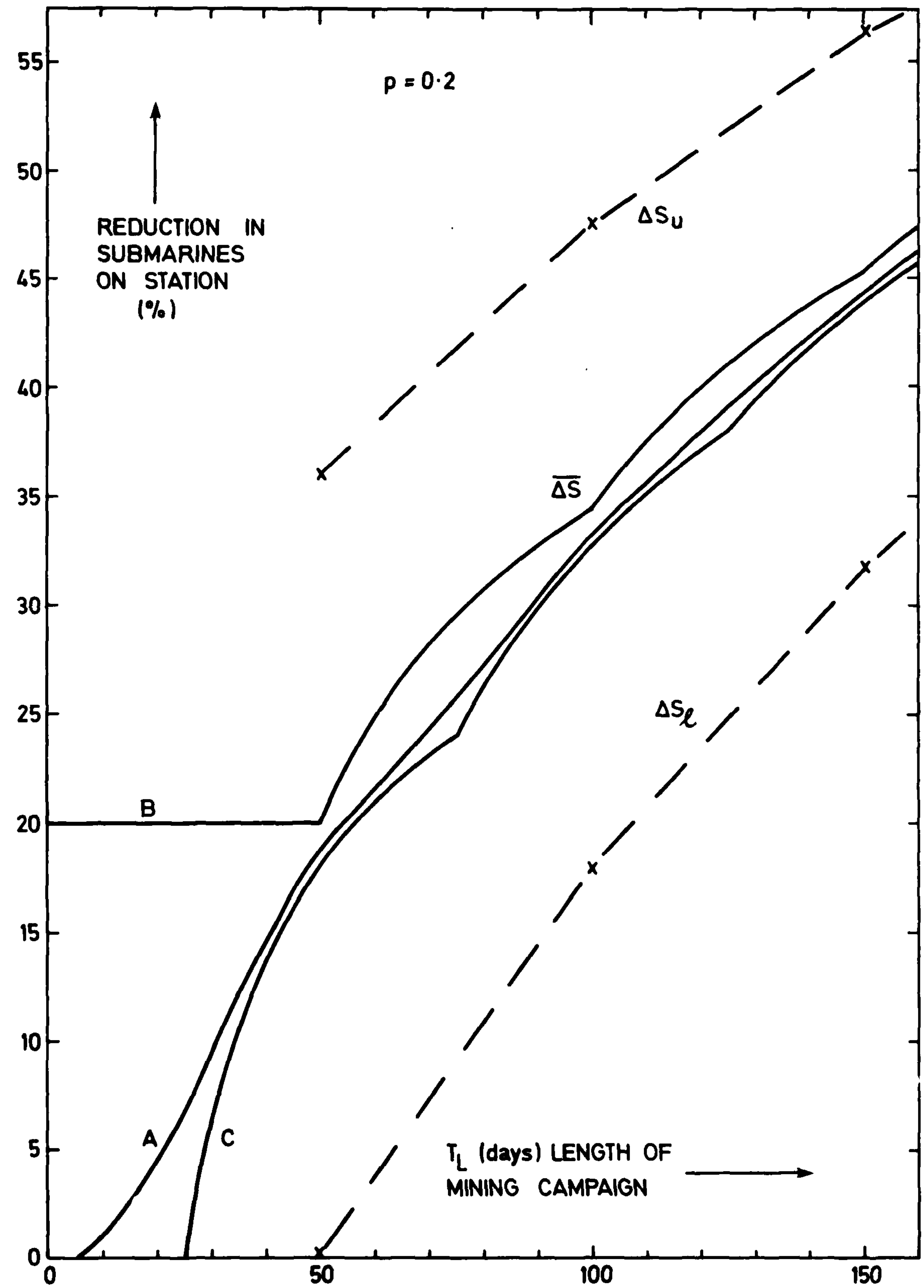

Fig. 3·1. Submarines traverse the minefield — dependence of submarine reduction on T_L.

large values of T_L, the actual situation (submarine operational cycle and minefield position) has little effect on the result for $\overline{\Delta S}$.

It is interesting to note that eqn. 3.11 simplifies considerably when the length of the mining campaign is an integral multiple of submarine operational cycles. When $T_L \doteq NT_{CP}$, $T_R = 0$ and the third term in eqn. 3.11 is zero; $\overline{\Delta S}$ is then given by

$$\overline{\Delta S} = 1 - \frac{1}{N}\left[q + \frac{(T_P + 2T_{T2})}{2T_{CP}}(1-q)^2\right]\left(\frac{1-q^{2N}}{1-q^2}\right). \qquad (3.14)$$

Furthermore, for the special cases $N = 1$ and $N = 2$, equations 3.12 to 3.14 become

$$\overline{\Delta S} = p - \frac{(T_P + 2T_{T2})}{2T_{CP}} \cdot p^2 \qquad (3.15)$$

$$\Delta S_\ell = 0 \qquad (3.16)$$

and $$\Delta S_u = 1 - q^2 \qquad (3.17)$$

for $N = 1$, and

$$\overline{\Delta S} = 1 - \frac{1}{2}\left[q + \frac{(T_P + 2T_{T2})}{2T_{CP}}(1-q)^2\right](1+q^2) \qquad (3.18)$$

$$\Delta S_\ell = \frac{1}{2}(1-q^2) \qquad (3.19)$$

and $$\Delta S_u = 1 - \frac{1}{2}q^2(1+q^2) \qquad (3.20)$$

for $N = 2$. Figure 3.2 shows results for $\overline{\Delta S}$, ΔS_ℓ, and ΔS_u as a function of p. It can be seen that for small values of p (which is often the case for practical minefields) the relationships are approximately linear. The lower and upper bounds (ΔS_ℓ and ΔS_u) are closer to the expected value ($\overline{\Delta S}$) when p is small and when N is large. Thus for low-risk minefields and long mining campaigns there is less uncertainty as to the actual reduction in submarines on station achieved.

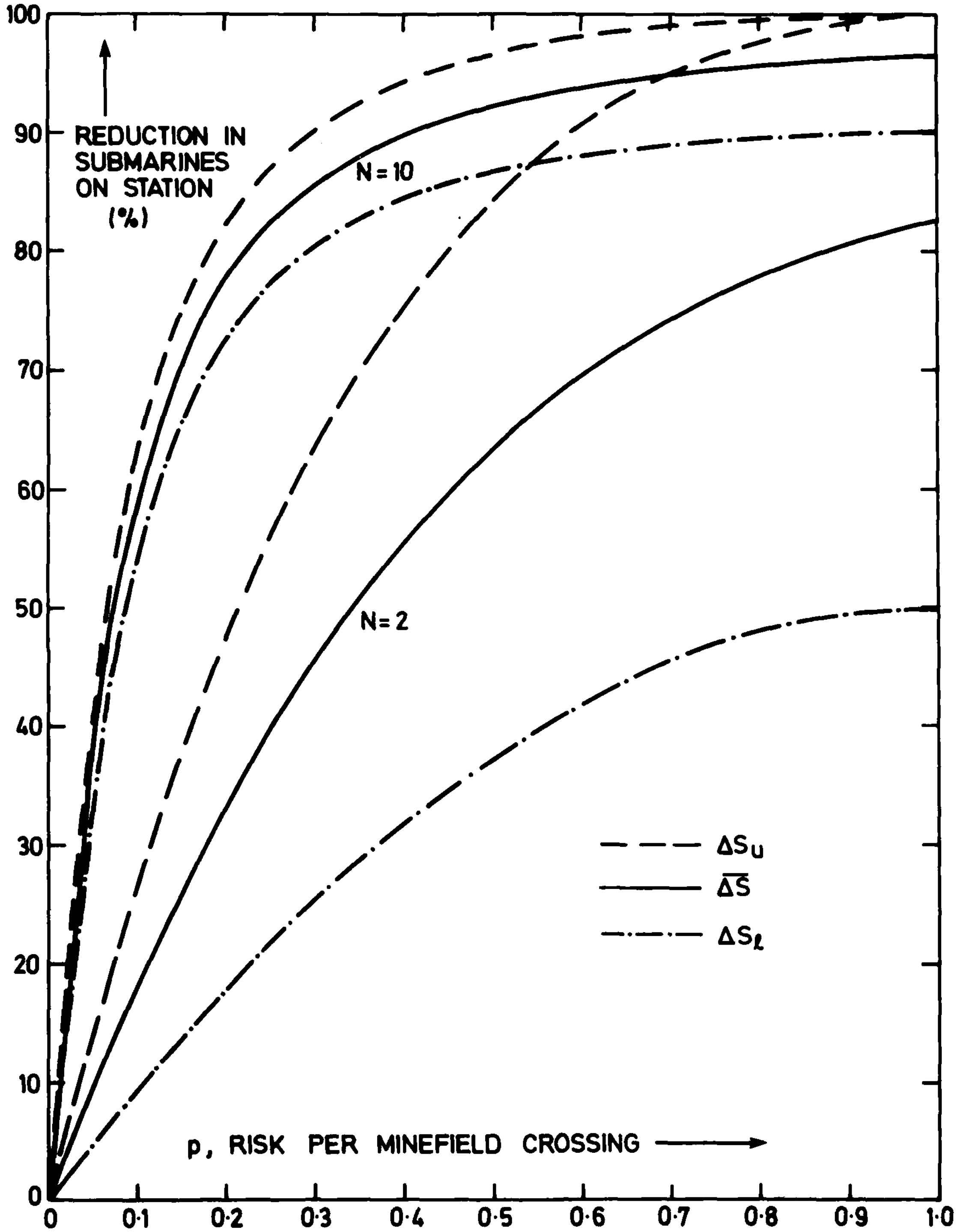

Fig. 3·2. Submarines traverse the minefield — dependence of submarine reduction on p.

Equation 3.14 (which applies only when $T_L = NT_{CP}$) can also be simplified if certain situations, involving the submarine operational cycle and the minefield location, are assumed. If $(T_P + 2T_{T2}) = 0$, then eqn 3.14 becomes

$$\overline{\Delta S} \quad = \quad 1 - \frac{q}{N} \left(\frac{1 - q^{2N}}{1 - q^2} \right) \tag{3.21}$$

This corresponds to situation B discussed previously (but for the special case $T_L = NT_{CP}$). If $(T_P + 2T_{T2}) = T_{CP}$, then eqn. 3.14 becomes

$$\overline{\Delta S} \quad = \quad 1 - \frac{(1 + q^2)}{2N} \cdot \left(\frac{1 - q^{2N}}{1 - q^2} \right) \tag{3.22}$$

This situation, denoted C', incorporates situation C discussed previously (i.e. $T_T = T_{T2} \approx T_{CP}$ and $T_P = 0$), but also includes the case where the minefield is at the approaches to the patrol area ($T_{T2} = 0$) and the submarines spend most of their time on station ($T_P = T_{CP}$). An intermediate situation, denoted D, occurs when $(T_P + 2T_{T2}) = T_{CP}/2$, the result obtained is

$$\overline{\Delta S} \quad = \quad 1 - \frac{(1 + q)^2}{4N} \cdot \left(\frac{1 - q^{2N}}{1 - q^2} \right) \tag{3.23}$$

which is the algebraic average of eqns. 3.21 and 3.22. Situations B, C' and D are shown in Fig. 3.3 where curves B and C' can be considered as extreme values for $\overline{\Delta S}$. It can be seen that the result $\overline{\Delta S}$ becomes less dependent on the particular situation when p is small and when N is large.

3.2.1 Simplifying Approximations

In view of the complexity of eqn. 3.11 there is a clear need to find simplifying approximations. As has already been shown, eqn. 3.11 simplifies considerably wherever $T_L = NT_{CP}$. Although $\overline{\Delta S}$ is strongly dependent on the value of T_L (see Fig. 3.1), the exact length of the mining campaign is rarely known in advance and in practical situations T_L is usually an assumed parameter. Therefore, it will often be acceptable to choose T_L such that $T_L = NT_{CP}$, so that eqn. 3.14 can be used instead of eqn. 3.11. (Note that if T_L is such that N = 1 or N = 2 eqns. 3.15 to 3.17, or eqns. 3.18 to 3.20 can be used.)

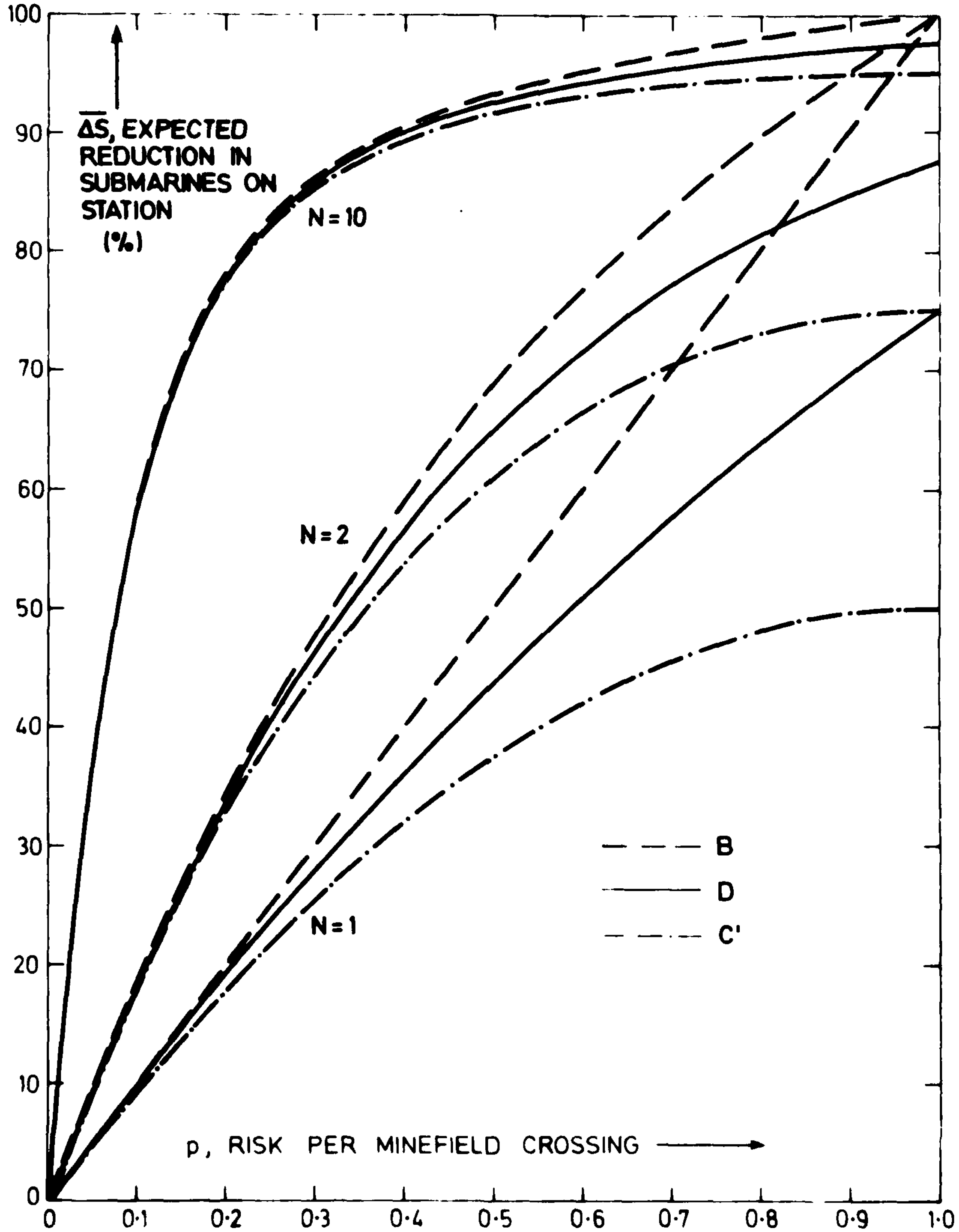

Fig. 3·3. Submarines traverse the minefield — **special cases.**

However, it is found that even when T_L is not an integral multiple of T_{CP}, eqn 3.14 is a good approximation to eqn. 3.11 provided that N (defined in eqn 3.2) is replaced by n' (eqn. 3.3), and that $n' \geq 1$. Similarly, N can be replaced by n' in eqns. 3.12 and 3.13 to determine lower and upper bounds. A comparison of "exact" and approximate results for $\overline{\Delta S}$, ΔS_ℓ, and ΔS_u for situation A (discussed previously) is shown in Fig. 3.4 (crosses indicate "exact" values of ΔS_ℓ and ΔS_u). It can be seen that for $n' \geq 1$, the agreement for $\overline{\Delta S}$ is quite good, and improves as T_L increases. The greatest discrepancy between exact and approximate solutions for $\overline{\Delta S}$ occurs for the "extreme" situations B and C in Fig. 3.1. It can also be seen that the approximate solutions for ΔS_ℓ and ΔS_u appear to be better approximations than the linear interpolation suggested in Section 3.2. Figure 3.4 clearly shows that the approximation deteriorates for $n' < 1$, particularly for ΔS_ℓ which becomes negative.

In addition to the two simplifying approximations discussed above, it may be acceptable to approximate a particular situation (submarine operational cycle, and minefield location) to that of situations B, C', or D, described in Section 3.2. The two sets of approximations may then be combined. Thus n' may be used with eqn 3.21 if $(T_P + 2T_{T2}) \simeq 0$, with eqn. 3.22 if $(T_P + 2T_{T2}) \simeq T_{CP}$, and with eqn. 3.23 if $(T_P + 2T_{T2}) \simeq T_{CP}/2$. As shown in Fig. 3.3, these approximations improve as p decreases and as N (or n') increases.

3.2.2 Long Term Model

The long term model is so called because it is assumed that the mining campaign is sufficiently long that the initial submarine position, when mining begins, has negligible effect on the final outcome. A simple long term model can be derived from calculus. ... rate of submarines is given by

$$\frac{dS}{dt} = -aS \qquad (3.24)$$

where S is the number of submarines, and a is a constant. Equation 3.24 shows that as the number of enemy submarines increases, more minefield crossings occur and therefore, the chance of a submarine casualty increases. The solution to eqn. 3.24 is

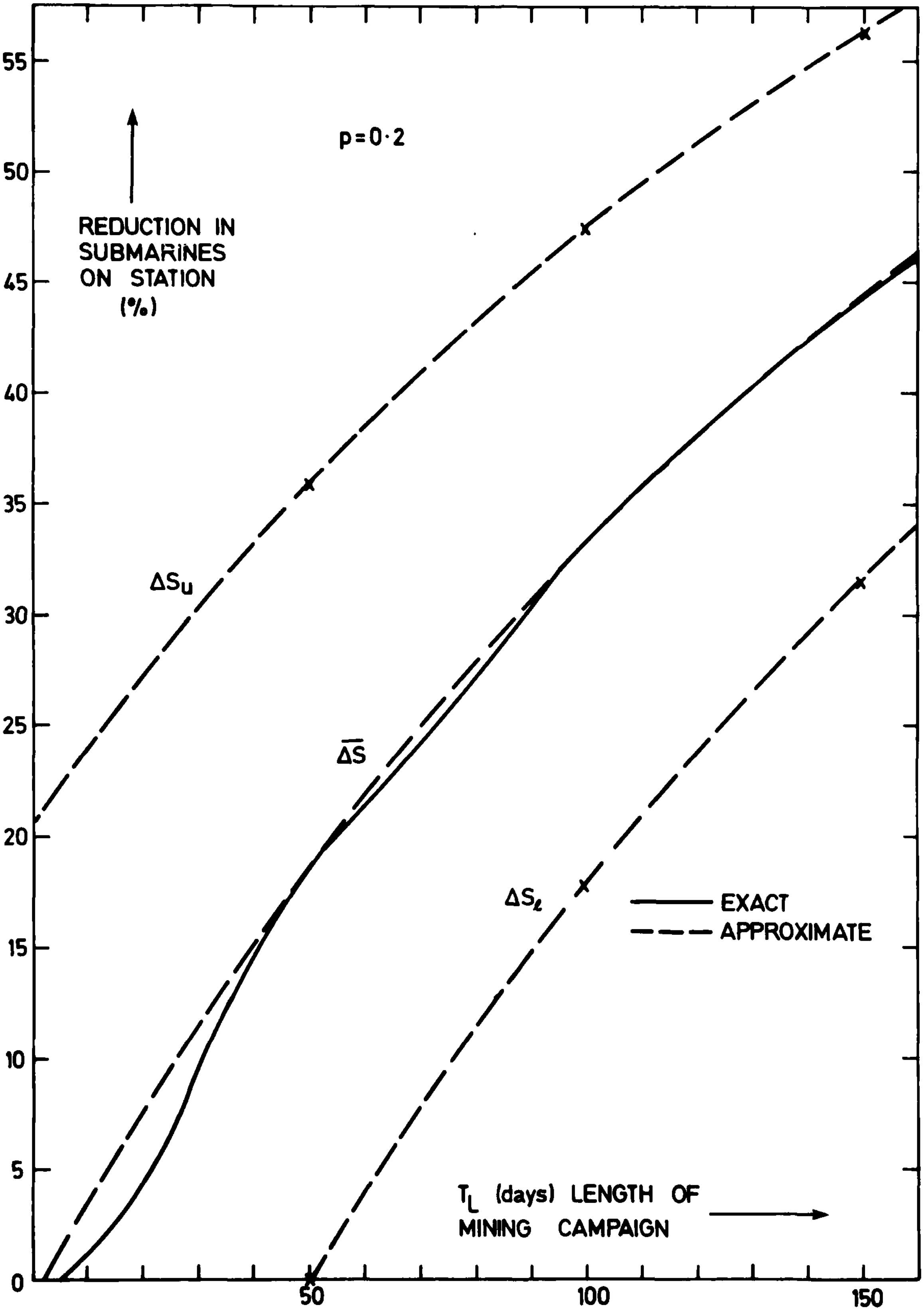

Fig. 3·4. Submarines traverse the minefield — comparison

$$S(t) = S_o \; e^{-at} \tag{3.25}$$

where S_o is the initial number of submarines at time $t = 0$. The fractional reduction in submarine numbers is given by

$$\Delta S(t) = \frac{S_o - S(t)}{S_o} = 1 - e^{-at} \tag{3.26}$$

The fractional reduction averaged over the length of the mining campaign is

$$\overline{\Delta S} = \frac{\int_o^{T_L} \Delta S(t)\, dt}{\int_o^{T_L} dt} \tag{3.27}$$

Solving,

$$\overline{\Delta S} = 1 - \frac{1}{aT_L} \left(1 - e^{-aT_L} \right) \quad . \tag{3.28}$$

In one submarine operational cycle $(t = T_{CP})$, there are two minefield crossings per submarine, and the expected number of surviving submarines is $S_o\,(1 - p)^2$. If this result is equated to that of eqn. 3.25, it is found that

$$a = \frac{-2}{T_{CP}} \; \ell n \; q \quad . \tag{3.29}$$

The solution for the long term model is obtained by substituting eqn. 3.29 for a into eqn. 3.28, thus

$$\overline{\Delta S} = 1 + \frac{T_{CP}}{2T_L \, \ell n \; q} \left(1 - q^{2T_L / T_{CP}} \right) \tag{3.30}$$

Figure 3.5 shows a comparison between long term model (eqn. 3.30) and probabilistic model solutions (eqns. 3.21 and 3.23). It can be seen that for small values of p, the long term model solution approaches that of situation D, defined in Section 3.2. For larger values of p the

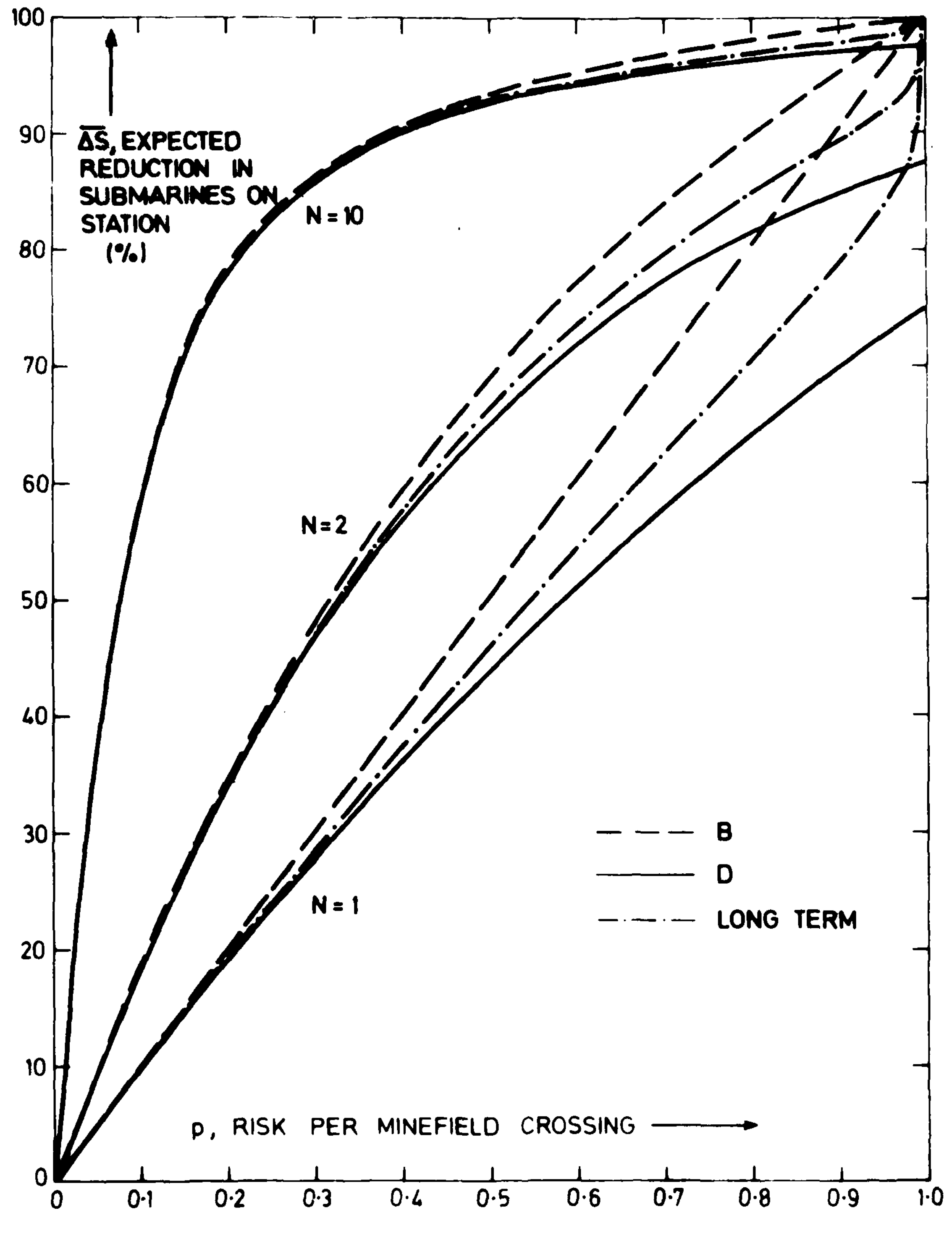

Fig. 3·5. Submarines traverse the minefield — comparison between long term model and probabilistic model solutions.

long term solution is between situations B and D. As expected, the agreement between the long term model and probabilistic model solutions improves as the length of the mining campaign increases (large N). It is also found that, as a function of T_L, the solution from eqn. 3.30 follows closely that of eqn. 3.14 with N replaced by n' (although a comparison is not shown, the dependence of eqn. 3.14 on T_L is shown in Fig. 3.4).

It can be concluded that although the simple model described above is adequate for small values of p and large values of T_L, it is considerably in error whenever p is large and/or T_L is small (particularly when $T_L < T_{CP}$).

3.3 MINEFIELD IS AVOIDED

The model described in this subsection represents the case where the submarines decide to take a detour around the minefield (assuming this is possible). The increase in the one-way transit time associated with the new route is denoted by ΔT. Since the submarine endurance is limited, i.e. T_{PL} is constant, the increase in transit time will reduce the time spent on station in the patrol area to

$$T_P' = T_P - 2\Delta T \qquad (3.31)$$

(This assumes that $2\Delta T \leq T_P$, otherwise the submarine never reaches the patrol area.) The new submarine operational cycle is described by eqn. A.4 in Annex A. Note that, in contrast to the model in Subsection 3.2, the number of occasions that the submarine is on station during the mining campaign remains the same.

3.3.1 Long Term Model

The long term model described here applies if the mining campaign is sufficiently long that the initial submarine position (in relation to its operational cycle) when the mining campaign begins, has negligible effect. Therefore, in the long term model the submarine is assumed to spend T_P' days in the patrol area each time it is on station. A submarine goes n' times on station (eqn. 3.3) during the mining campaign, but each such occasion is only a fraction T_P'/T_P of its former length. Mathematically it is simpler to solve for the

equivalent (in this model) situation where a submarine spends T_P days on station but where the number of occasions it is on station is reduced to

$$E(n) = n' \cdot \frac{T_P'}{T_P} \qquad (3.32)$$

The long term model solution is obtained by substituting this value of E(n) into eqn. 3.4, thus,

$$\overline{\Delta S} = 1 - \frac{T_P'}{T_P}$$

i.e.
$$\overline{\Delta S} = \frac{2\Delta T}{T_P} \qquad (3.33)$$

It can be seen that $\overline{\Delta S}$ is linear in ΔT .

3.3.2 General Model

The general model described here is based on the concept outlined for the long term model (Subsection 3.3.1). However, in this instance, the initial submarine position (within its operational cycle) when the mining campaign begins, is taken into account. For example, if the submarine is at its base when mining begins, it has to take the detour on its first transit and the time on station is shortened by $2\Delta T$ days; on the other hand, if it has just arrived on station, it will only have to take the detour on its return journey, and the time on station is only shortened by ΔT days.

In the model described here, it is assumed that the additional transit time, ΔT, due to the detour is constant so that the total one-way transit time assumes a constant value

$$T_{T1} + \Delta T + T_{T2} \qquad (3.34)$$

In reality, the value of ΔT on the first transit should depend on the submarine position at the instant the mining campaign begins. For example, if a submarine is at its base when minelaying begins in a strait, it is able to determine the shortest detour necessary to reach

the patrol area, and the increase in transit time is ΔT. On the other hand, if a submarine is approaching the strait when minelaying begins, it may have to backtrack part of the way in order to take the detour, and the increase in transit time, $\Delta T'$, will be greater than ΔT. Note that $\Delta T'$ applies only to the first transit, on all subsequent transits the increa~e in transit time is ΔT. For this reason, $\Delta T'$ is expected to have a negligible effect on the final result ($\overline{\Delta S}$, ΔS_ℓ, and ΔS_u), particularly for a lengthy mining campaign. A more complex mathematical model (not described here), which takes $\Delta T'$ into account, has confirmed this expectation when applied to a number of practical situations.

An additional problem arises if the mining campaign begins just as a submarine has completed its time on station. In this case, the submarine will have spent T_P days on station, but must take the detour on its return journey and as a result the total patrol length becomes

$$2T_T + T_P + \Delta T \quad . \tag{3.35}$$

It can be seen that the submarine endurance or patrol length, T_{PL}, is exceeded by ΔT days (Annex A). In the model presented in this section the length of the submarine operational cycle, T_{CP}, is kept constant by shortening the length of time spent at the submarine base to $T_B' = T_B - \Delta T$. This approximation corresponds to a situation where the enemy is able to hasten submarine replenishment so that submarines leave their base on time. It must be emphasised that the shortened time in port, T_B', applies only after the first patrol; after subsequent patrols submarines spend T_B days at their base.

The expected reduction in submarines on station (derived in Annex C) is given by

$$\overline{\Delta S} = \frac{\Delta T}{T_P} \qquad T_L \leq T_{T2} \tag{3.36}$$

$$= \frac{\Delta T}{T_P} + \frac{1}{2.T_L.T_P} \cdot (T_L - T_{T2})^2 \qquad T_{T2} < T_L < T_{T2} + \Delta T$$

$$= \frac{2\Delta T}{T_P} - \frac{\Delta T}{T_L.T_P} \left(T_{T2} + \frac{\Delta T}{2}\right) \qquad T_L \geq T_{T2} + \Delta T$$

with the conditions that

$$\Delta T \leq T_B \quad \text{and} \quad \Delta T \leq T_P/2$$

(A more complex model, where the constraint $\Delta T \leq T_B$ is not necessary, is available at RANRL on request.) The lower and upper bounds are given by

$$\Delta S_\ell = \frac{2\Delta T}{T_P} \left(1 - \frac{1}{2N} \right) \tag{3.37}$$

and $$\Delta S_u = \frac{2\Delta T}{T_P} \tag{3.38}$$

It is interesting to note that the upper bound, ΔS_u, is equal to the result for the long term model (eqn 3.33). Equations 3.37 and 3.38 apply to the case where $T_L = NT_{CP}$; interpolation is necessary for other values of T_L. The best interpolation for ΔS_ℓ is to substitute n' (eqn. 3.3) for N in eqn. 3.37. Clearly, a linear interpolation applies for ΔS_u.

Figure 3.6 shows results for $\overline{\Delta S}$, ΔS_ℓ, and ΔS_u for case A, described in Subsection 3.2, ($T_{CP} = 50$, $T_B = 5$, $T_P = 25$, and $T_{T1} = T_{T2} = 5$ days) with $\Delta T = 3$ days. Crosses indicate values of ΔS_ℓ and ΔS_u shere $T_L = NT_{CP}$ is satisfied. As expected $\overline{\Delta S}$ approaches the long term model value as T_L increases. It can be seen that in the long term, a 30% increase in transit time will reduce the number of submarines on station by 24% (for this particular case).

If T_L is sufficiently large, eqn. 3.36 may be approximated by the long term value (eqn. 3.33), and $\overline{\Delta S}$ is linearly dependent on ΔT. (The constraint that $\Delta T \leq T_B$ is then no longer necessary.) Since eqn. 3.36 is relatively simple to use, other simplifying approximations are not necessary.

3.4 MINEFIELD IS CLEARED

The model described in this subsection represents the case where the submarines decide to wait until the minefield has been cleared before proceeding on their transit. The waiting time for submarines for each transit is denoted by T_W. If the minefield is located at a strait, it is expected that a submarine would wait at a nearby port and would

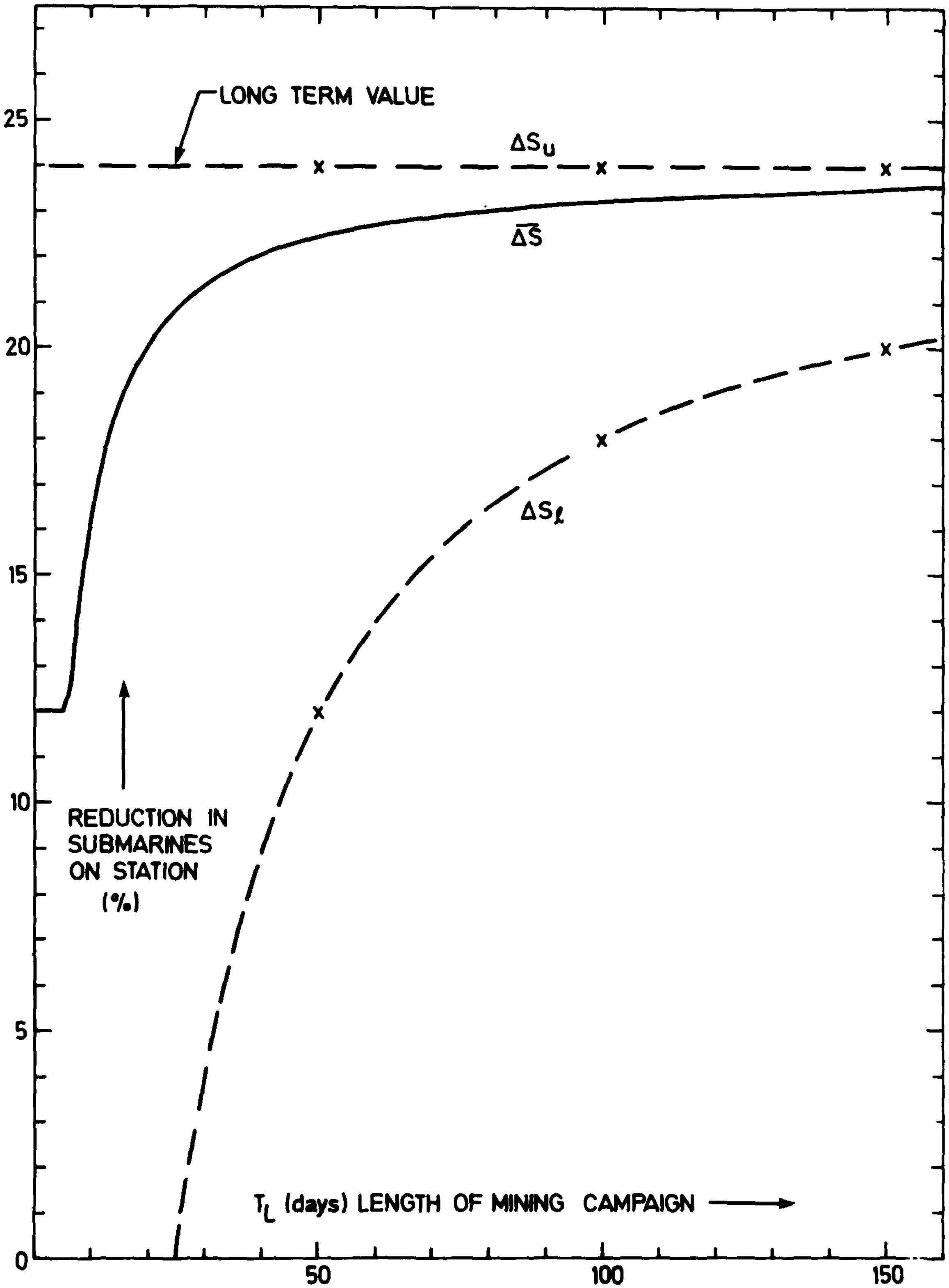

Fig. 3·6. Submarines go around the minefield — dependence of submarine reduction on T_L.

thus be able to replenish some of its stores. Similarly, if the minefield is located at the submarine base and a submarine is outside the base, it is expected that some limited replenishment could take place. However, in either case it is assumed that a submarine must return to its base to reload weapons and carry out repairs.

Since some limited replenishment takes place while a submarine waits for minefield clearance, the submarine endurance is increased to $T_{PL} + 2T_W$ and the time spent on station in the patrol area, T_P , remains the same. The time between consecutive patrol is increased to

$$T_{CP}' = T_{CP} + 2T_W \tag{3.39}$$

and, as a consequence, over a given length of time (T_L) a submarine goes on fewer patrols than if mining had not taken place. Thus, even though T_P is unchanged, there is a reduction in the average number of submarines on station.

Since the waiting time T_W essentially determines the reduction in submarines on station, it is of interest to relate it to the actual time, T_C , taken to clear the minefield. It can be shown (Annex D) that if submarines arrive at the minefield at random, and if the time between consecutive minefield replenishments is T_{RP} , then the expected waiting time experienced by the submarines is

$$T_W = \frac{T_C^{\,2}}{2T_{RP}} \tag{3.40}$$

with standard deviation

$$\sigma_W = T_W \cdot \left[\frac{4T_{RP}}{3\,T_C} - 1 \right]^{\frac{1}{2}} . \tag{3.41}$$

3.4.1 Long Term Model

The long term model described here applies if the mining campaign is sufficiently long that the initial submarine position (in relation to its operational cycle) when mining begins has negligible effect. Therefore, the time between consecutive submarine patrols is

taken to be that given by eqn. 3.39. The expected number of times a submarine is on station during the minimg campaign is

$$E(n) = \frac{T_L}{T_{CP} + 2T_W} \tag{3.42}$$

Substitution of eqns. 3.42 and 3.3 into eqn. 3.4 gives

$$\overline{\Delta S} = \frac{2T_W}{T_{CP} + 2T_W} \tag{3.43}$$

for the long term model solution. This result is plotted in Fig. 3.7 as a function of the ratio T_W/T_{CP}. It can be seen that a significant reduction in submarines on station ($\overline{\Delta S} \simeq 0.3$) can be achieved for relatively small values of T_W/T_{CP} ($T_W/T_{CP} \simeq 0.2$). However, large values of T_W/T_{CP} are subject to the law of diminishing returns; $\overline{\Delta S}$ approaches unity as T_W/T_{CP} approaches infinity.

3.4.2 General Model

The general model described here takes account of the initial submarine position (with its operational cycle), when mining begins. This initial position determines the delay experienced by the submarine during the first cycle (see Fig. D.1 in Annex D). This delay is either 0, T_W, or $2T_W$; for all subsequent cycles the delay is $2T_W$. The derivation for this model is given in Annex D.

The expected reduction in submarines on station is given by

$$\overline{\Delta S} = \frac{2N'T_W}{T_L} + \frac{1}{T_L \cdot T_P}\Big[G(T_{CP}' - T_P - T_{T2}, T_W, T_P) + G(T_{T2} + T_W, T_W, T_P)\Big] \tag{3.44}$$

where $$N' = \text{Int}\left[\frac{T_L}{T_{CP} + 2T_W}\right] \tag{3.45}$$

and the function G(y,z,m) is defined by eqn F.3 in Annex F. The value of T_R used in evaluating G(y,z,m) is

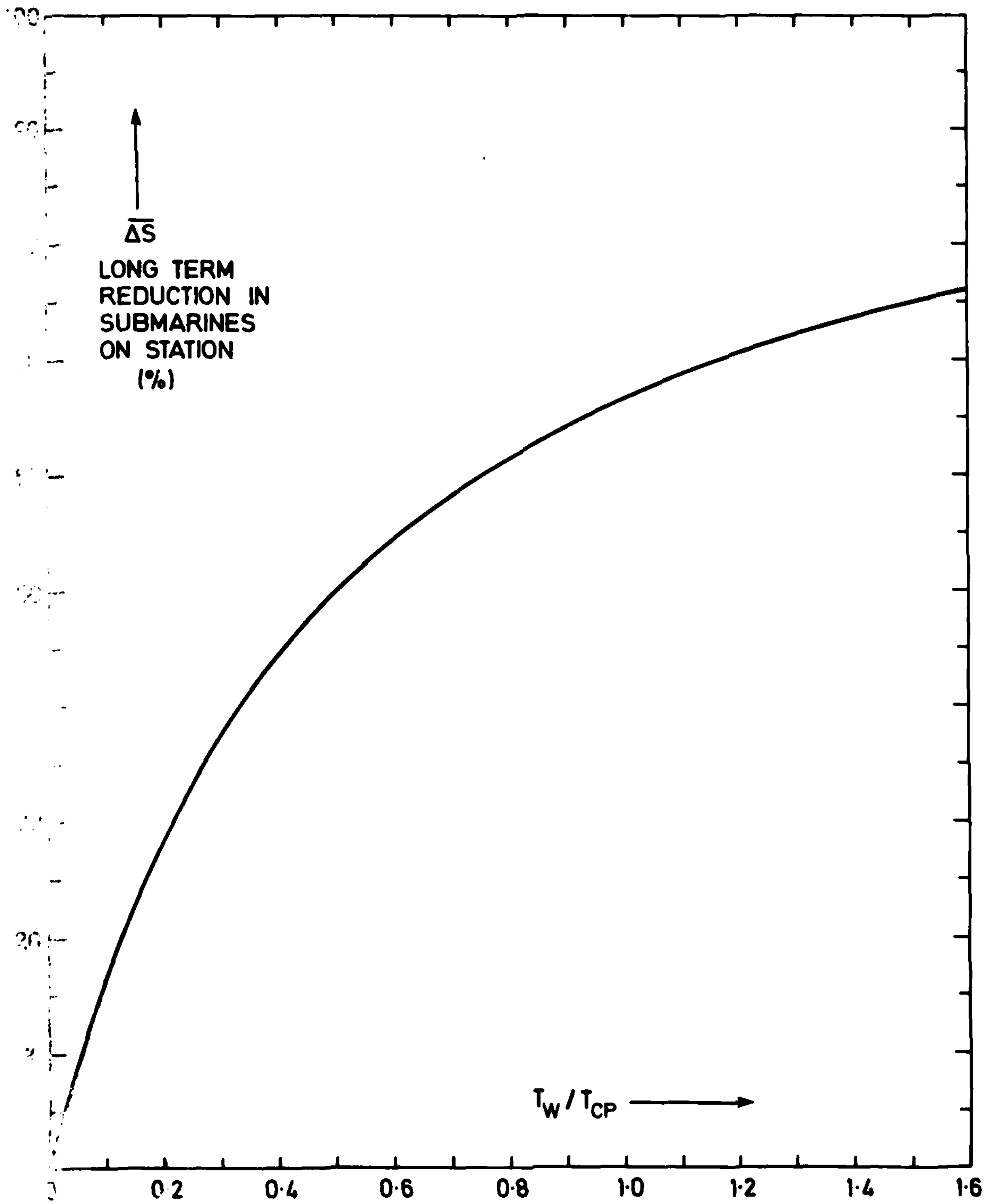

Fig. 3·7. Minefield is cleared — long term model.

$$T_R' \quad = \quad T_L - N(T_{CP} + 2T_W) \quad . \tag{3.46}$$

Note that when $T_R' = 0$, the second term in eqn. 3.44 is zero. Thus when $T_L = N'T_{CP}'$, eqn. 3.44 is equal to the long term value (eqn. 3.43).

Lower and upper bounds (which apply only when $T_L = NT_{CP}$) are given by

$$\Delta S_\ell \quad = \quad 1 - \frac{N'}{N} - \frac{1}{N} \; . \; \mathrm{Min} \left[\; 1 \; ; \; \frac{T_R'}{T_P} \right] \tag{3.47}$$

and

$$\Delta S_u \quad = \quad 1 - \frac{N'}{N} - \frac{1}{N.T_P} \; . \; \mathrm{Max} \left[0; \; T_R' + T_P - T_{CP}' \right] \tag{3.48}$$

A typical result is shown in Fig. 3.8 for $T_W = 3$ (i.e. $T_W/T_{CP} = 0.06$). This corresponds to a situation similar to case A in Subsection 3.2 except that the minefield is located at the submarine base ($T_B = 5$, $T_{T1} = 0$, $T_{T2} = 10$, $T_P = 25$, $T_{CP} = 50$ days). It can be seen that $\overline{\Delta S}$ oscillates *about the long term* value (eqn. 3.43), the amplitude of oscillation decreasing as T_L becomes large. The oscillations can be understood by examining eqn. 3.44. When $T_L = N'T_{CP}'$, $T_R' = 0$ and $\overline{\Delta S}$ is equal to the long term value. In the range

$$N'T_{CP}' \quad \leq \quad T_L \quad < \quad (N' + 1)T_{CP}'$$

N' remains constant and the first term in eqn. 3.44 decreases as T_L^{-1} . In this range of values for T_L, T_R' will be in the range

$$0 \quad \leq \quad T_R' \quad \leq \quad T_{CP}' \qquad .$$

The sum of the functions $G(y,z,m)$ in the second term in equ. 3.44 is zero for $T_R' \leq T_{T2}$ and increases to reach a constant value $2T_W$ when $T_{CP}' - T_{T2} \leq T_R' \leq T_{CP}'$. The net result is that $\overline{\Delta S}$ decreases in the range $0 \leq T_R' \leq T_{T2}$, increases in the range $T_{T2} \leq T_R' \leq T_{CP}' - T_{T2}$, and decreases in the range $T_{CP}' - T_{T2} \leq T_R \leq T_{CP}'$. These successive increases and decreases give rise to oscillations, with a "wavelength" of T_{CP}', in the value of $\overline{\Delta S}$ as T_L increases. The damping of these oscillations is explained by rearranging eqn. 3.44,

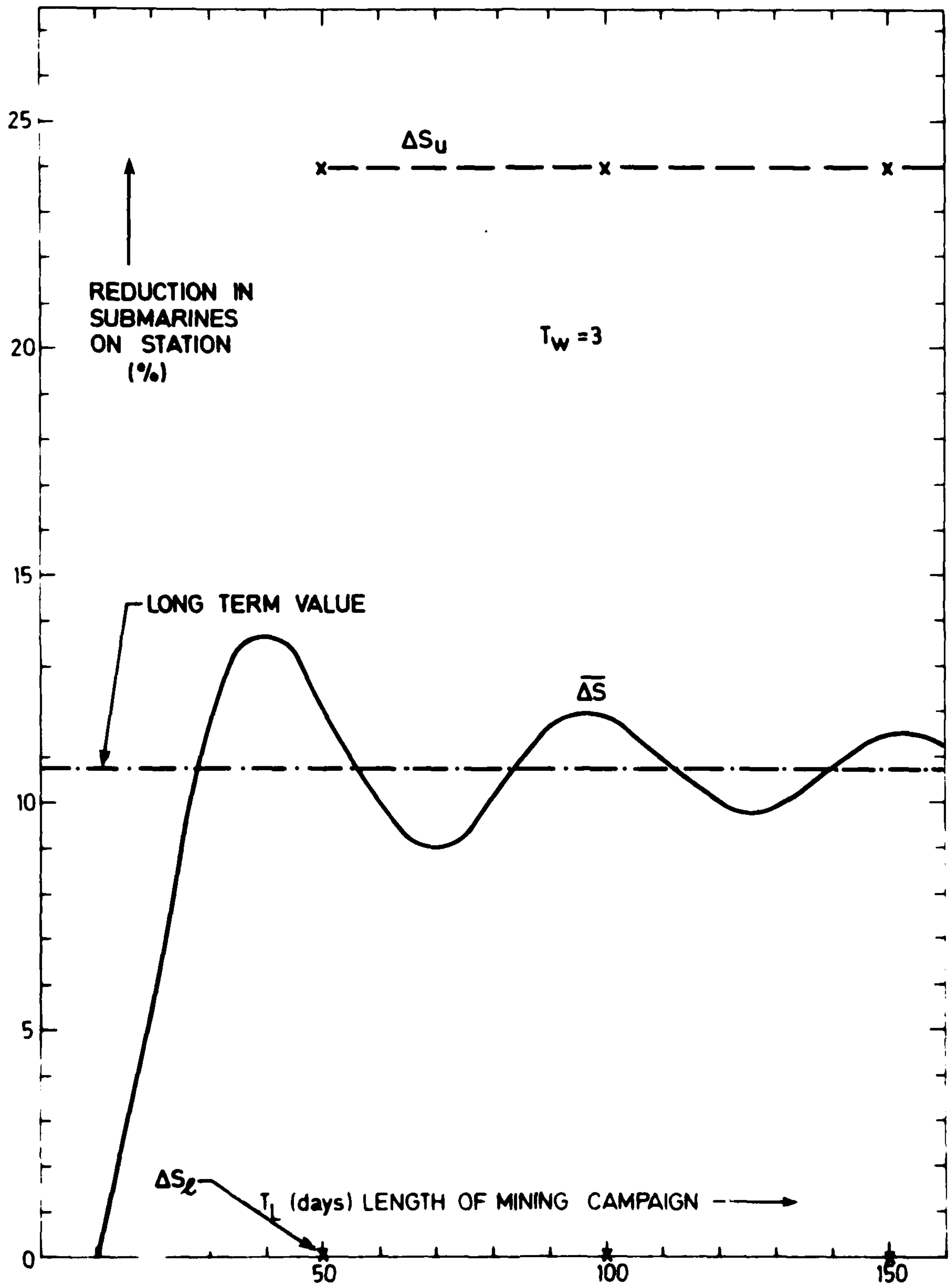

Fig. 3·8. Minefield is cleared — dependence of submarine reduction on T_L.

$$\overline{\Delta S} = \frac{2T_W}{T_{CP}'} + \frac{1}{T_L} \left\{ \frac{1}{T_P} (G_1 + G_2) - \frac{2T_W T_R'}{T_{CP}'} \right\} \qquad (3.49)$$

where G_1 and G_2 represent the functions $G(y,z,m)$ in eqn. 3.44. Equation 3.49 shows that $\overline{\Delta S}$ is equal to the long term value, plus a second term which accounts for the oscillations and which decreases as T_L^{-1}. Since for all values of T_L,

$$0 \leq G_1 + G_2 \leq 2T_W T_P$$

and $$0 \leq T_R' \leq T_{CP}'$$

the range of values in the brackets in eqn. 3.49 is constant. Therefore, the effect of the second term, and hence the oscillations, decreases as T_L increases.

The lower and upper bounds, ΔS_ℓ and ΔS_u, which appear to be constant in Fig. 3.8, do vary for larger values of T_L. The variation of ΔS_ℓ and ΔS_u as a function of T_L is shown in Fig. 3.9. Note that eqns. 3.47 and 3.48 are only valid for $T_L = NT_{CP}$; these values of ΔS_ℓ and ΔS_u are indicated by crosses in Fig. 3.9 (the dashed lines are merely a guide to the eye). It can be seen that the values of both ΔS_ℓ and ΔS_u oscillate; these oscillations are damped and approach the long term value (eqn. 3.43) for large T_L. These oscillations ar caused by variations in T_R' which governs the second term in eqn. 3.47 and 3.48. The difference between ΔS_ℓ and ΔS_u is smallest whenever both T_{CP} and T_{CP}' are common factors of T_L, i.e. whenever $T_R \simeq T_R' \simeq 0$.

3.4.3 Model Simplification

As has been already shown in the previous subsection, when T_L is an integral multiple of T_{CP}' (i.e. $T_L = N'T_{CP}'$) the second term in eqn. 3.44 is zero and $\overline{\Delta S}$ is equal to the long term value (eqn. 3.43). Thus by choosing appropriate values of T_L, eqn. 3.44 can be simplified considerably.

When T_L is not an integral multiple of T_{CP}', eqn. 3.44 may be approximated by removing the dependence on T_R' (i.e. by eliminating the second term) and by replacing N' by n'', where $n'' = T_L/T_{CP}'$. (This approximation is essentially the same as that outlined in Subsection

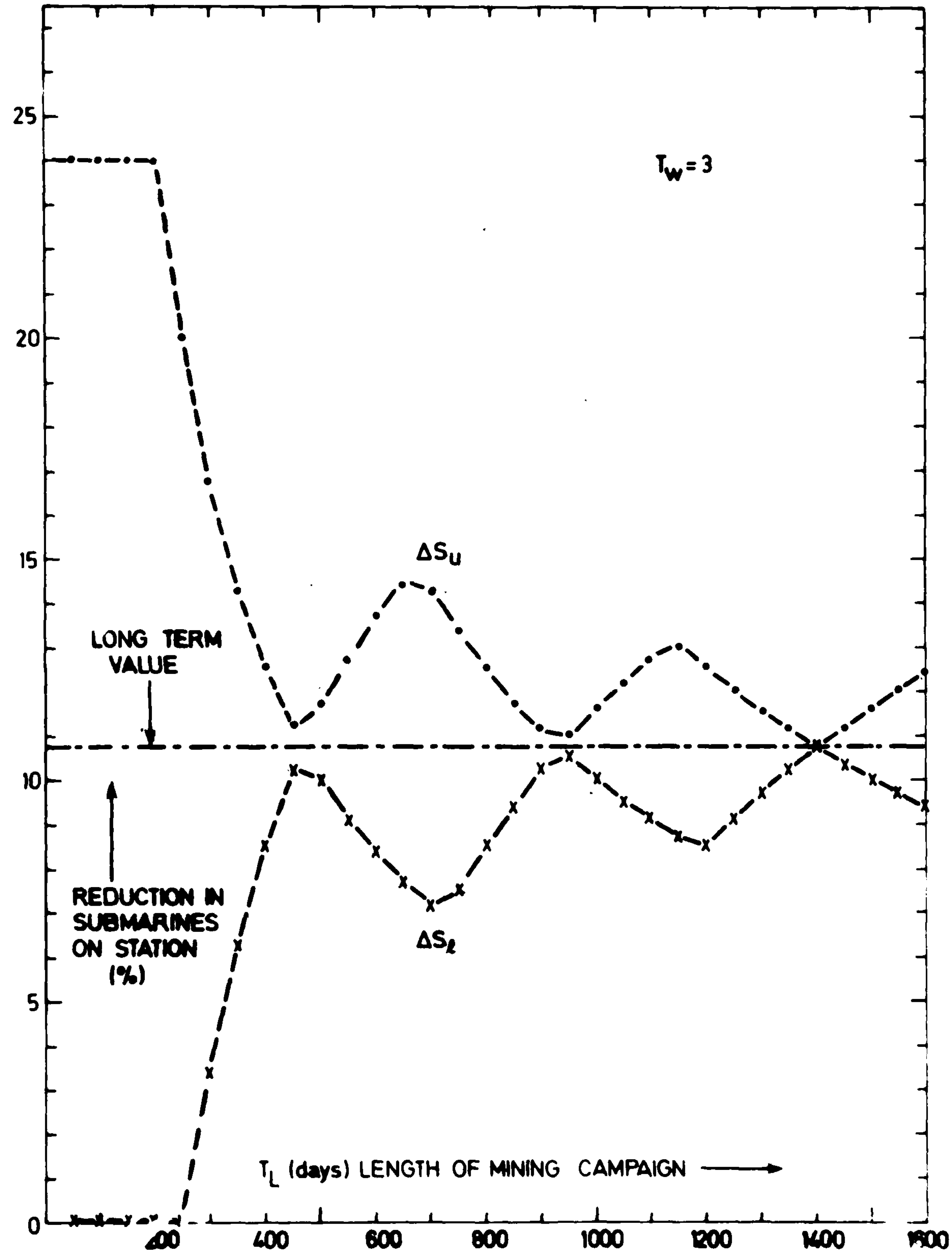

Fig. 3·9. Minefield is cleared — dependence of ΔS_ℓ and ΔS_u on T_L.

3.2.1). However, when this approach is followed, it is found that the approximate solution to eqn. 3.44 is equal to the long term value. Figure 3.8 shows that, in general, the long term value is a reasonable approximation to $\overline{\Delta S}$ only for large values of T_L.

The above approximations simplify the dependence of $\overline{\Delta S}$ on T_L. However, in many practical ca‹es, T_L is fixed and the dependence of $\overline{\Delta S}$ on T_W is more important. It can be shown (Annex D) that for $T_L = NT_{CP}$, eqn. 3.44 simplifies to

$$\overline{\Delta S} = \frac{2T_W}{T_{CP}} \cdot \left[1 - \frac{(n-1)}{N} \right] \tag{3.50}$$

provided that T_W is in the range(s),

$$\frac{(n-1)\,T_{CP} - T_{T2}}{2\left[N - (n-1)\right]} \leq T_W \leq \frac{(n-1)\,T_{CP} + T_{T2}}{2\left[N - (n-1)\right]} \tag{3.51}$$

where $n = 1, 2, \ldots, N$ and N is given by eqn. 3.2. Note that eqn. 3.50 is an exact solution for $\overline{\Delta S}$ and not an approximation. It can be seen that if condition 3.51 is satisfied then $\overline{\Delta S}$ is linear in T_W. To test condition 3.51 it may be necessary to evaluate both sides of the inequality for all possible values of n.

Since $n = N - N'$ (see Annex D), for short mining campaigns the most commonly occurring value is $n = 1$. For this special case eqn. 3.50 becomes

$$\overline{\Delta S} = \frac{2T_W}{T_{CP}} \tag{3.53}$$

with the condition that

$$T_W \leq \frac{T_{T2}}{2N} \tag{3.53}$$

In terms of the time, T_C, taken to clear the minefield, condition 3.53 becomes (using eqn. 3.40)

$$T_C \leq \left[\frac{T_{T2}\, T_{RP}}{N} \right]^{\frac{1}{2}} \qquad (3.54)$$

If T_{RP}, the time between consecutive minefield replenishments, is not known, then the worst case $T_{RP} = T_C$ must be assumed and the necessary condition for eqn. 3.52 to be satisfied becomes

$$T_C \leq \frac{T_{T2}}{N} \qquad (3.55)$$

4. MINING OPTIONS

The mathematical models described in Section 3 are developed in terms of the course of action taken by a submarine when confronted by a minefield. However, it is necessary to consider the types of minefields that can be laid by the minelaying nation, as these will influence the policy adopted by the submarines. The four mining options considered here are

a. a single minelay
b. intermittent mining
c. attrition mining
d. covert mining

Each of these options is discussed below.

4.1 SINGLE MINELAY

This mining option consists of laying an initial minefield, either at the submarine base or in a strait, with no further minefield replenishment during the remainder of the war. This form of mining is expected to be effective, in the strategic sense, only if the enemy has no MCM forces, or if these forces are engaged elsewhere. Thus, if attrition mining (see Subsection 4.3) is being carried out at the submarine base and the enemy MCM forces are fully occupied there, it may be possible to effectively mine a strait with a single minelay and no further replenishment.

The effectiveness of this type of minefield is dependent on the traffic being low, so that mine depletion (by being actuated) does not significantly affect p , the level of risk. The effectiveness of the minefield also depends on minefield planning such that actuation of

a single wide radius of action mine does not leave a known "gap" in the minefield.

4.2 INTERMITTENT MINING

This form of mining consists of an initial minelay to blockade a port or strait, followed by periodic replenishments of the minefield to compensate for enemy MCM efforts. The replenishments are relatively infrequent and the MCM forces are able to clear a channel through the minefield before the next replenishment.

The submarines have the option of traversing the minefield, waiting for minefield clearance, or, where possible, going around the minefield. In the case where the submarines decide to traverse the minefield, the level of risk, p, to be used in the mathematical model described in Subsection 3.2, is given by

$$p = p_M \cdot \left(\frac{T_C}{T_{RP}} \right) \tag{4.1}$$

where p_M is the probability of a casualty per minefield crossing, given that a channel has not been cleared. (Delayed arming and ship count can be used to make p_M approximately constant during most of the clearance effort, with a sharp drop-off occurring at the very end.) The second factor in eqn. 4.1 represents the proportion of the time that a channel is not clear.

4.3 ATTRITION MINING

Attrition mining is similar to intermittent mining except that the minefield replenishments are sufficiently frequent that the enemy MCM forces never succeed in establishing a cleared channel. Use of delayed arming and ship count may also be made to achieve this end. The risk of a submarine casualty per minefield crossing, p, is assumed to be constant in time.

Since a channel through the minefield is never cleared, the mathematical model described in Subsection 3.4 may not be used. The only submarine option is to traverse the minefield, or, in the case of straits, to go around it.

4.4 COVERT MINING

To avoid detection, a covert minefield would normally be laid by submarine. Only an initial minelay is necessary. Such a minefield is practicable only where there are no MCM vessels on task and the field remains undetected. It must be noted that covert mining would normally require contravention of the Hague Convention (1907) and therefore, may not be feasible.

Submarines are expected to continue to transit through the minefield until its discovery. The effectiveness of a covert minefield, measured by $\widetilde{\Delta S}$, depends strongly on the time of its discovery. In general, $\overline{\Delta S}$ will lie between two extremes:

a. If a covert minefield at a strait is discovered early in the war, the submarines will go around it and $\overline{\Delta S}$ is given by the mathematical model described in Subsection 3.3. (If the covert minefield is at the submarine base, then the submarines would wait for a channel to be cleared. Since only a single minelay is involved, the effect of the delay is negligible in the strategic sense.)

b. If the minefield is discovered late in the war, submarines will have continuously traversed the minefield for most of the war and $\widetilde{\Delta S}$ is given by the model in Subsection 3.2.

In the absence of other information being available to the enemy (and assuming that the mines are not detonated by surface shipping), it may be assumed that minefield discovery occurs with the first submarine casualty. (Even then, it may not be clear to the enemy that the submarine casualty is due to mines). It may be shown (Annex E) that the expected time, T_D, for the first submarine casualty is

$$T_D = \frac{T_{CP}}{2pS} \qquad (4.2)$$

where S is the number of submarines regularly transiting through the covert minefield. The standard deviation for T_D is

$$\sigma_D = T_D \sqrt{1 - p} \qquad (4.3)$$

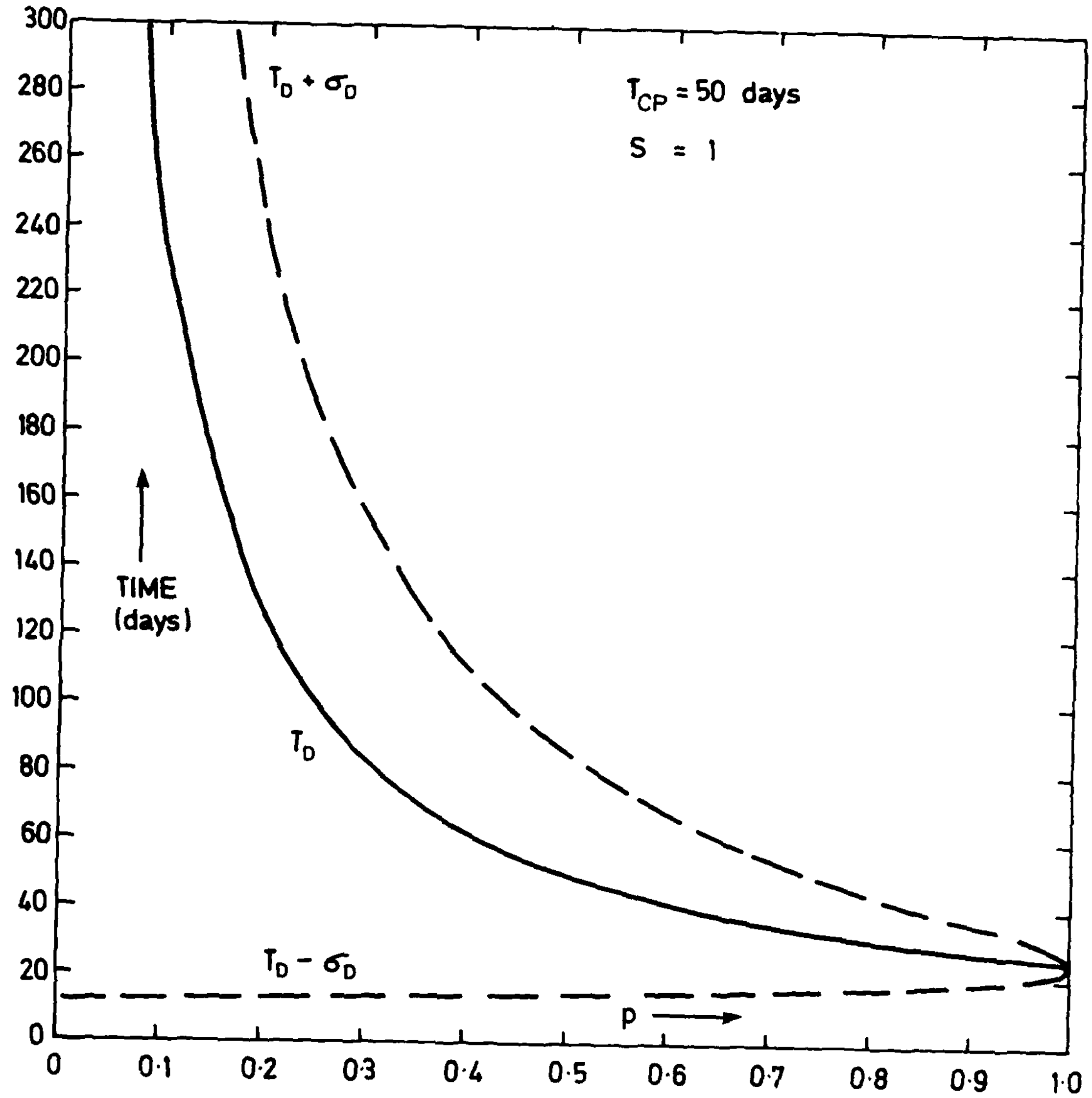

Fig. 4·1. Expected time for first submarine casualty.

Figure 4.1 shows results for $T_D - \sigma_D$, T_D , and $T_D + \sigma_D$ as a function of p for the case where $S = 1$ and $T_{CP} = 50$ days. It can be seen that for large values of p (i.e. $p \to 1$) the standard deviation, σ_D , is small, and minefield discovery is expected to occur early in the war (i.e. T_D is small); this corresponds the case a. above. For small values of p (i.e. $p < 0.3$) the expected discovery time, T_D , increases but the standard deviation, σ_D , is so large that minefield discovery could occur at almost any time during the war. Therefore, for small values of p , T_D is not a useful estimate of the minefield discovery time. In this situation, the best estimate of $\overline{\Delta S}$ that can be made is that it lies between the two extreme cases, a. and b. described above, i.e. $\overline{\Delta S}$ lies between $\overline{\Delta S}_a$ and $\overline{\Delta S}_b$.

5. OPTIMAL MINING POLICY

The optimal mining policy, for the minelayer, is that which maximises the return, ΔS_ℓ , $\overline{\Delta S}$, and ΔS_u , in spite of the enemy submarines' attempts to minimise it. A decision theory approach is used in this section to determine the optimal mining policy.

The options (described in Sections 3 and 4) available to the two opponents may be combined as shown in Table 5.1. The entries in this table are $\overline{\Delta S}$, the expected reduction in submarines on patrol, for each mining policy - submarine policy combination. Although not shown in this table, consideration should also be given to ΔS_ℓ and ΔS_u when assessing alternatives. It must also be noted that not all options in Table 5.1 are always feasible; for example, if the submarine base is mined, it is not normally possible for submarines to avoid the minefield by going around it.

In finding the optimal policies, it is first necessary to determine the required level of risk, p , for each mining option. For example, consider the case where the submarine base is mined intermittently. If p is small, there is little likelihood of a casualty and the best submarine policy is to traverse the minefield. However, once p exceeds a certain value, the danger becomes too great and the submarine will wait for a channel to be cleared. Any larger value of p will result in unnecessary mine numbers since the channel clearance time remains constant (this applies to minesweeping given that the mine settings such as ship court and delayed arriving are unchanged). Clearly, the optimum value of

TABLE 5.1

POLICY OPTIONS FOR PORTS AND STRAITS

		MINING POLICY			
		SINGLE MINELAY	INTERMITTENT	ATTRITION	COVERT
SUBMARINE POLICY	TRAVERSE MINEFIELD	$\overline{\Delta S}_1$	$\overline{\Delta S}_4$	$\overline{\Delta S}_7$	$\overline{\Delta S}_9$
	AVOID MINEFIELD	$\overline{\Delta S}_2$	$\overline{\Delta S}_5$	$\overline{\Delta S}_8$	
	CLEAR MINEFIELD	$\overline{\Delta S}_3$	$\overline{\Delta S}_6$		

p is such that

$$\overline{\Delta S_4} = \overline{\Delta S_6} \tag{5.1}$$

In practice, p is obtained by solving the above equation numerically. Similarly, in the case where a strait is mined intermittently the optimum level of mining is determined by finding p such that

$$\overline{\Delta S_4} = \text{Min} \left[\overline{\Delta S_5} \; ; \; \overline{\Delta S_6} \right] \tag{5.2}$$

If p is too small, the submarines will traverse the minefield. If p is sufficiently large, the submarines will either wait for clearance or take a detour, whichever gives the lesser $\overline{\Delta S}$.

The above method for determining the optimum p assumes that the submarines has precise information about the minefield. In a practical situation, the enemy will not know the precise value of p , although he may be able to estimate it by observing the minelaying operation or from the results of MCM efforts. The tendency will be for submarines to err on the side of safety and not traverse the minefield (the psychological effect of mine warfare is also important here). If such is the case, a smaller value of p than that prescribed above may achieve the same result.

Once the optimum value of p , and hence the maximum value of $\overline{\Delta S}$, has been determined for each mining option, the solution of the optimization problem for mining can be found. The best overall mining option is that which gives the largest value of $\overline{\Delta S}$.

Once the optimum mining policies have been determined for both ports and straits, it is necessary to decide whether to mine the submarine base only, the straits only, or both. As a first step, the results ($\overline{\Delta S}$, ΔS_ℓ, and ΔS_u) for the best mining policy for mining the submarine base or the straits can be compared to see which gives the better return. However, this return must normally be related to the cost of the mining campaign, and mining the submarine base may be more cost-effective even though the return ($\overline{\Delta S}$) from mining straits is higher.

If mining both the submarine base and the straits is under

in the optimisation problem for mining. For example, consider the case where the enemy has a small MCM capability, and attrition mining of the base or attrition mining of the straits are the best individual mining policies. If both the base and the straits are mined, the enemy may decide to keep the entire MCM force engaged in clearing the submarine base; if such is the case, a single minelay in the straits would suffice. Thus, in this example, the combined mining cost is less than the sum of the individual mining costs of the base and the strait. If both the submarine base and the straits are mined, the overall return is given by

$$\overline{\Delta S}\ (\text{overall}) = 1 - \left[1 - \overline{\Delta S}\ (\text{base})\right] . \left[1 - \overline{\Delta S}\ (\text{straits})\right]. \quad (5.1)$$

6. CONCLUSION

In this note, mathematical models have been developed to assess the effectiveness of a strategic offensive ASW mining campaign in reducing the average number of submarines that an enemy can maintain on station in a patrol area. This reduction in the average number of submarines on station is due to damage by mines, detours taken to avoid minefields, and/or time lost waiting for minefields to be cleared. The tactical effects of mining are not assessed. The principal assumptions incorporated into these models are that the enemy submarines operate in a regular cycle between their base and a patrol area, and that once damaged or sunk they cannot be repaired or replaced during the mining campaign. Within these constraints, the models are quite general.

A special feature of the models presented here is that they can be applied to very short mining campaigns. It has therefore been possible to test the validity of simpler models based on the assumption of long term mining operations. In all three sets of models, it is found that there is a significant discrepancy between the "exact" expectation value and the long term value whenever the mining campaign is short. This is particularly so when $T_L < T_{CP}$. However, for long mining campaigns ($T_L >> T_{CP}$), the two values approach one another asymptotically.

The results obtained from the mathematical models depend on the values of all input parameters but are particularly sensitive to the probability of a submarine casualty per minefield crossing (p) ,

the additional transit time when taking a detour (ΔT), and the expected time that a submarine must wait for minefield clearance (T_W). In most practical mine warfare situations however, there is considerable uncertainty in the values of the input parameters and hence a consequent uncertainty in the results obtained from the models. The complexity of some "exact" models may therefore be unnecessary. For this reason, simplifying approximations have been developed for each case. Also, if the mining campaign is reasonably long, it may be sufficient to use the long term model prediction rather than the more complicated "exact" solution.

When assessing the potential effectiveness of a mining campaign, it is not sufficient to consider the expected return, $\overline{\Delta S}$, only; the range of possible returns, specified by ΔS_ℓ and ΔS_u, must also be taken into account. Thus, one mining policy may be preferred over another because there is less variability in the potential return (i.e. the difference between ΔS_ℓ and ΔS_u is small) even though the expected return, $\overline{\Delta S}$, is smaller. In general, the difference between ΔS_ℓ and ΔS_u decreases as the length of the mining campaign increases.

The optimal mining policy, i.e. the type of mining and the level (p) of mining, can be determined by using a decision theory approach. Although the method outlined in Section 5 tends to give an overestimate for the optimum level of mining (p), this value can serve as a useful guide for estimating a realistic value of p.

Although only simple situations have been treated here, the mathematical models described in this note, together with the decision theory approach, can form a useful basis for estimating the requirements and results of a strategic ASW mining campaign in more complex studies.

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ANNEX A

SUBMARINE OPERATIONAL CYCLE

The mathematical models developed here are based on a well-defined submarine operational cycle. This cycle is illustrated in Fig. A.1. The cycle length is T_{CP} (time between consecutive patrols) and is subdivided into a number of time periods:

T_B = time spent by submarines at their base

T_{T1} = transit time between submarine base and minefield

T_{T2} = transit time between minefield and patrol area

T_T = total one-way transit time

T_P = time spent on station in the patrol area

T_{PL} = submarine patrol length

Note that,

$$T_T = T_{T1} + T_{T2} ,$$

$$T_{PL} = T_P + 2T_T ,$$

$$T_{CP} = T_B + T_{PL} \quad (A.1)$$

and

$$T_{CP} = T_B + T_P + 2T_T \quad (A.2)$$

These equations and the above parameters are for the case where no mining has occurred.

If the submarines follow a policy of waiting until the minefield is cleared, then the total one-way transit time is

$$T_{T1} + T_W + T_{T2} = T_T + T_W$$

where T_W is the waiting time. The operational cycle is lengthened to

$$T_{CP}' = T_B + T_P + 2T_T + 2T_W = T_{CP} + 2T_W \qquad (A.3)$$

If the submarines follow a policy of going around the minefield then the one-way transit time is

$$T_{T1} + \Delta T + T_{T2} = T_T + \Delta T$$

where ΔT is the additional transit time due to the detour taken. The time spent in the patrol area is reduced to

$$T_P' = T_P - 2\Delta T$$

and the operational cycle is more easily visualised by

$$T_{CP} = T_B + T_P' + 2T_T + 2\Delta T \qquad (A.4)$$

(this reduces to eqn. A.2).

If the submarines follow a policy of traversing the minefield, then the operational cycle is unchanged from that in eqn. A.2.

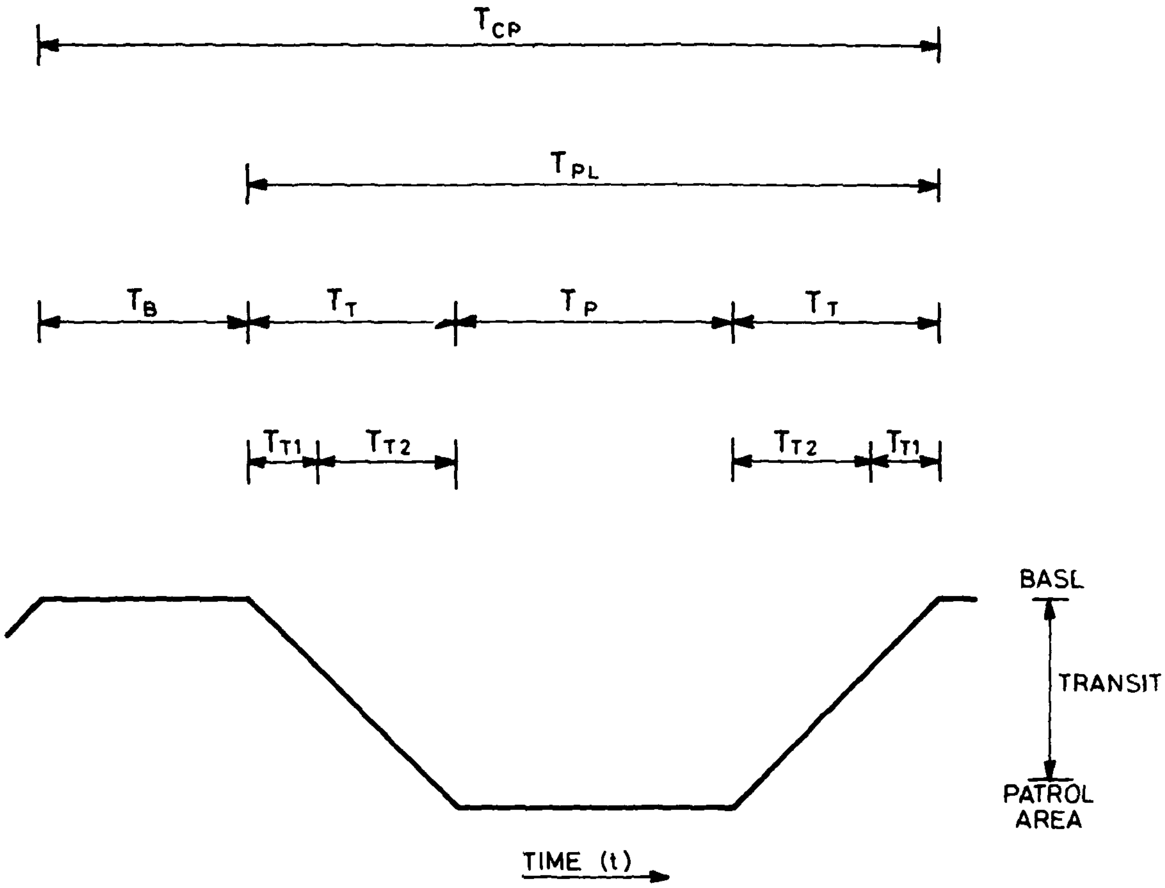

Fig. A·1. Submarine operational cycle.

ANNEX B

MINEFIELD IS TRAVERSED - DERIVATION OF MODEL

The concept of this model is described in Subsection 3.2. For simplicity, eqn. 3.14 which applies for the special case when $T_L = NT_{CP}$ is derived first; the more general case (eqn. 3.11), where T_L can assume any value, is derived in Section B.3.

B.1 CASE WHERE $T_L = NT_{CP}$

The general procedure used to solve this case is to determine P(n patrols) and P(N patrols) from eqns. 3.8 and 3.10, and to solve for E(n) using eqn. 3.9. This procedure is simplified by subdividing the submarine operational cycle (Annex A) into four parts, depending on the submarine position when the mining campaign begins (see Fig. B.1):

a. The submarine has not yet crossed the minelaying area on its way to the patrol area. The number of times that this submarine will be in the patrol area will depend on which minefield crossing is being attempted when it is sunk, this is shown in Table B.1.

TABLE B.1

Crossing when sunk (i)	Number of times on patrol
1	0
2 or 3	1
4 or 5	2
.	.
.	.
.	.
2n or 2n + 1	n

The probability of exactly n patrols (eqn. 3.8) is

$$P(\text{n patrols}) = Pr(2n) + Pr(2n+1)$$

$$= pq^{2n-1} + pq^{2n} \qquad \text{(B.1)}$$

The probability of exactly N patrols (eqn. 3.10) is

$$P(\text{N patrols}) = 1 - \sum_{n=0}^{N-1} \left[Pr(2n) + Pr(2n + 1) \right]$$

$$= 1 - \sum_{i=1}^{2N-1} Pr(i) \qquad \text{taking } Pr(0) = 0$$

$$= q^{2N-1} \qquad (B.2)$$

The substitution of eqns. B.1 and B.2 into eqn 3.9 gives

$$E(n)_a = q \left(\frac{1 - q^{2N}}{1 - q^2} \right) \qquad (B.3)$$

(In solving for eqn. B.3, use can be made of the solution for series no. 5 in Ref. 2).

b. The submarine has already crossed the minefield area on its way to the patrol area. The solution is obtained in a similar way to that in part a. but using Table B.2.

TABLE B.2

Crossing when sunk (i)	Number of times on patrol
1 or 2	1
3 or 4	2
5 or 6	3
.	.
.	.
.	.
2n-1 or 2n	n

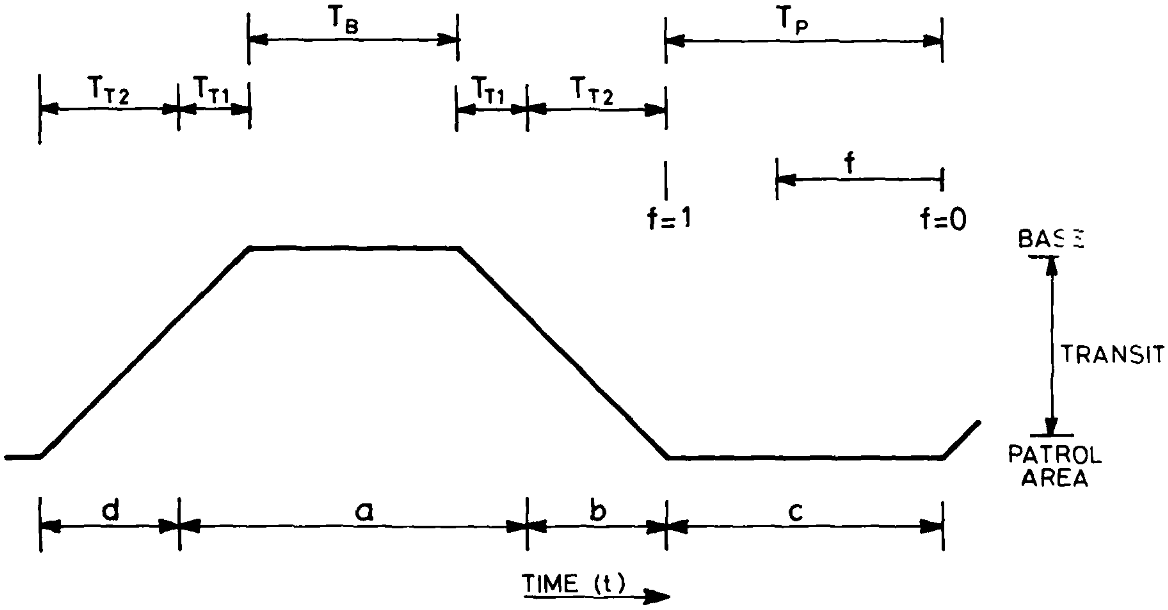

Fig. B-1. Partitioning of submarine operational cycle used in model derivation.

$$P(n \text{ patrols}) = Pr(2n - 1) + Pr(2n)$$

$$P(N \text{ patrols}) = q^{2N-2}$$

$$E(n)_b = \left(\frac{1 - q^{2N}}{1 - q^2} \right) \qquad (B.4)$$

c. The submarine is somewhere on patrol. In this part it is necessary to take into account f, the fraction of the time to be spent on station which remains to be completed $(0 \leq f \leq 1)$. The number of times on patrol are shown in Table B.3.

TABLE B.3

Crossing when sunk (i)	Number of times on patrol
1 or 2	f
3 or 4	1 + f
5 or 6	2 + f
.	.
.	.
.	.
2n+1 or 2n+2	n + f

Equations 3.8 and 3.10 become

$$P(n + f \text{ patrols}) = Pr(2n + 1) + Pr(2n + 2) \qquad (B.5)$$

and $$P(N \text{ patrols}) = q^{2N} \qquad (B.6)$$

Equations 3.9 must be modified to

$$E(n) = N\, P(N \text{ patrols}) + \sum_{n=0}^{N-1} (n + f)\, P(n + f \text{ patrols}) \qquad (B.7)$$

The substitution of eqns. B.5 and B.6 into eqn. B.7 gives

To take account of all possibilities in this part, it is necessary to average over all values of f ,

$$E(n)_c = \int_0^1 \left[f + (1-f)q^2 \right] \left(\frac{1-q^{2N}}{1-q^2} \right) df$$

i.e. $$E(n)_c = \frac{1}{2}(1+q^2) \cdot \left(\frac{1-q^{2N}}{1-q^2} \right) . \qquad (B.9)$$

d. The submarine has just completed a patrol but has not yet crossed the minefield area on its way to its base. The solution is similar to parts a. and b.

TABLE B.4

Crossing when sunk (i)	Number of times on patrol
1 or 2	0
3 or 4	1
5 or 6	2
.	.
.	.
.	.
2n+1 or 2n+2	n

$$P(n \text{ patrols}) = Pr(2n+1) + Pr(2n+2)$$

$$P(N \text{ patrols}) = q^{2N}$$

$$E(n)_d = q^2 \left(\frac{1-q^{2N}}{1-q^2} \right) \qquad (B.10)$$

The overall expected number of times on station, E(n) , is obtained by combining the results of the four parts above, according to their frequency of occurrence (see Fig. B.1). Thus E(n) is given by

$$E(n) = \frac{(T_B + 2T_{T1})}{T_{CP}} E(n)_a + \frac{T_{T2}}{T_{CP}} E(n)_b + \frac{T_P}{T_{CP}} E(n)_c + \frac{T_{T2}}{T_{CP}} E(n)_d . \quad (B.11)$$

The solution to eqn. B.11 is

$$E(n) = \left[q + \frac{(T_P + 2T_{T2})}{2T_{CP}} (1 - q)^2 \right] \left(\frac{1 - q^{2N}}{1 - q^2} \right) . \quad (B.12)$$

Equation 3.14 is obtained from eqns. B.12, 3.3, and 3.4.

B.2 LOWER AND UPPER BOUNDS

Examination of parts a. to d. in Section B.1 shows that E(n) is a minimum in part d. and is a maximum in part b. Therefore, the lower and upper bounds for the number of times a submarine spends on station are given, respectively, by

$$n_\ell = q^2 \left(\frac{1 - q^{2N}}{1 - q^2} \right) \quad (B.13)$$

and

$$n_u = \left(\frac{1 - q^{2N}}{1 - q^2} \right) \quad (B.14)$$

Results for ΔS_ℓ and ΔS_u (eqns. 3.12 and 3.13) are obtained by substituting eqns. B.14 and B.13 into eqns. 3.5 and 3.6, respectively, assuming $T_L = NT_{CP}$.

B.3 CASE WHERE T_L ASSUMES ANY VALUE

The procedure to derive eqn. 3.11 is similar to that followed in Section B.1, except that account must be taken of T_R (defined in eqn. 3.1), the remaining number of days after all complete submarine operational cycles have been subtracted from T_L . Equation 3.9 is modified to

$$E(n) = \left[N + F(y, z) \right] P(N + 1 \text{ patrols}) + \sum_{n=1}^{N} n \; P(n \text{ patrols}) \quad (B.15)$$

$$\text{where} \quad F(y, z) = \frac{G(y, z, T_P)}{z.T_P} , \quad 0 \leq F(y, z) \leq 1 \quad (B.16)$$

is the expected additional fraction of time spent on station due to T_R . The function $G(y, z, m)$ is defined in eqn F.3.

As in Section B.1, it is convenient to divide the problem into four parts. Note that Tables B.1 and B.4 still apply, and the results for P(n patrols) are identical and are not repeated here. The four parts are discussed below:

a. Here,

$$P(N + 1 \text{ patrols}) = q^{2N+1} , \tag{B.17}$$

$$y = T_B + 2T_{T1} + T_{T2} , \quad \text{and} \quad z = T_B + 2T_{T1} .$$

The solution to eqn. B.15 is

$$E(n)_a = q\left(\frac{1 - q^{2N}}{1 - q^2}\right) + q^{2N+1} \cdot \frac{G(T_B + 2T_{T1} + T_{T2}, T_B + 2T_{T1}, T_P)}{T_P \cdot (T_B + 2T_{T1})} \tag{B.18}$$

b. Here,

$$P(N + 1 \text{ patrols}) = q^{2N} , \tag{B.19}$$

$$y = T_{T2} , \quad \text{and} \quad z = T_{T2} .$$

The solution to eqn. B.15 is

$$E(n)_b = \left(\frac{1 - q^{2N}}{1 - q^2}\right) + q^{2N} \cdot \frac{G(T_{T2}, T_{T2}, T_P)}{T_P \cdot T_{T2}} \tag{B.20}$$

c. For this part eqn. B.15 must be modified to

$$\begin{aligned} E(n) = & \left[N + f + F(T_{CP}, T_P)\right] P(N + 1 \text{ patrols}) \\ & + \left[N + F(0,T_P) - (1 - f)\right] \left[\Pr(2N + 1) + \Pr(2N + 2)\right] \\ & + \sum_{n=0}^{N-1} (n + f)\, P(n + f \text{ patrols}) . \end{aligned} \tag{B.21}$$

Here,

$$P(N + 1 \text{ patrols}) = q^{2N+2} , \tag{B.22}$$

and the solution to eqn. B.21 is

$$E(n) = \left[f + (1 - f)q^2 \right] . \left(\frac{1 - q^{2N}}{1 - q^2} \right) + q^{2N} \left[\frac{G(0,T_P,T_P)}{T_P{}^2} - (1 - f) \right]$$

$$+ q^{2N+2} . \frac{G(T_{CP},T_P,T_P)}{T_P{}^2} . \tag{B.23}$$

Integration over f gives,

$$E(n)_c = \frac{1}{2} (1 + q^2) . \left(\frac{1 - q^{2N}}{1 - q^2} \right) + q^{2N} \left[\frac{G(0,T_P,T_P)}{T_P{}^2} - \frac{1}{2} \right]$$

$$+ q^{2N+2} \frac{G(T_{CP},T_P,T_P)}{T_P{}^2} . \tag{B.24}$$

d. Here,

$$P(N + 1 \text{ patrols}) = q^{2N+2} , \tag{B.25}$$

$$y = T_{CP} - T_P , \text{ and } z = T_{T2} .$$

The solution to eqn. B.15 is

$$E(n)_d = q^2 \left(\frac{1 - q^{2N}}{1 - q^2} \right) + q^{2N+2} . \frac{G(T_{CP} - T_P, T_{T2}, T_P)}{T_P . T_{T2}} \tag{B.26}$$

The overall expected number of times on station, E(n), is obtained by substituting eqns. B.18, B.20, B.24 and B.26 into eqn. B.11. The result is

$$E(n) = \left[q + \frac{(T_P + 2T_{T2})}{2T_{CP}} (1 - q)^2 \right] . \left(\frac{1 - q^{2N}}{1 - q^2} \right)$$

$$+ \frac{q^{2N}}{T_P T_{CP}} \left\{ G(T_{T2},T_{T2},T_P) + G(0,T_P,T_P) - \frac{1}{2} T_P^{\,2} \right.$$

$$+ q \cdot G(T_B + 2T_{T1} + T_{T2}\ ,\ T_B + 2T_{T1},\ T_P)\ q^{2N+1}$$

$$\left. + q^2 \left[G(T_{CP}.T_P,T_P) + G(T_{CP} - T_P,\ T_{T2},\ T_P) \right] \right\} \qquad (B.27)$$

Equation 3.11 is obtained by substituting eqns. B.27 and 3.3 into eqn. 3.4.

ANNEX C

MINEFIELD IS AVOIDED - DERIVATION OF MODEL

The concept of this model is described in Subsection 3.3. The solution for the expected value, $\overline{\Delta S}$, (eqn. 3.36) is derived in Section C.1; lower and upper bounds (eqns. 3.37 and 3.38) are derived in Section C.2.

C.1 EXPECTED VALUE

The solution for the expected value, $\overline{\Delta S}$, is derived here for the general case where T_L can assume any value. The procedure adopted here, is to subdivide the submarine operational cycle (Annex A) into seven parts, depending on the submarine position (within its operational cycle) when the mining campaign begins. For each part, E(n), the equivalent number of occasions a submarine is on station in time T_L (similar in concept to that in eqn. 3.32) is evaluated. In each expression for E(n), the first term represents the contribution from the time period NT_{CP} , and the second term the contribution from T_R . Each of the seven parts is indicated in Fig. C.1 and the solutions for E(n) are given below.

a. Here, mining has begun before the submarine has passed the minefield area. The submarine must therefore, take a detour on its first patrol as well as all subsequent ones. The solution is

$$E(n)_a = N\frac{T_P'}{T_P} + \frac{G(T_B + T_T + \Delta T,\ T_B + T_{T1},\ T_P')}{T_P.(T_B + T_{T1})} \qquad (C.1)$$

where T_P' is defined in eqn. 3.31, and the function G(y,z,m) is defined by eqn. F.3. (Note that use of eqn. F.2 has been made in the second term.)

b. Here, mining has begun after the submarine has passed the minefield area. A detour on the first transit is not necessary, and the first occasion on station is of length $T_P' + \Delta T$ days. The solution is

$$E(n)_b = \frac{(NT_P' + \Delta T)}{T_P} + \frac{1}{T_P}\left[\frac{1}{T_{T2}} . G(T_{T2} + \Delta T,\ T_{T2}, T_P') + Z(N)_b\right] \qquad (C.2)$$

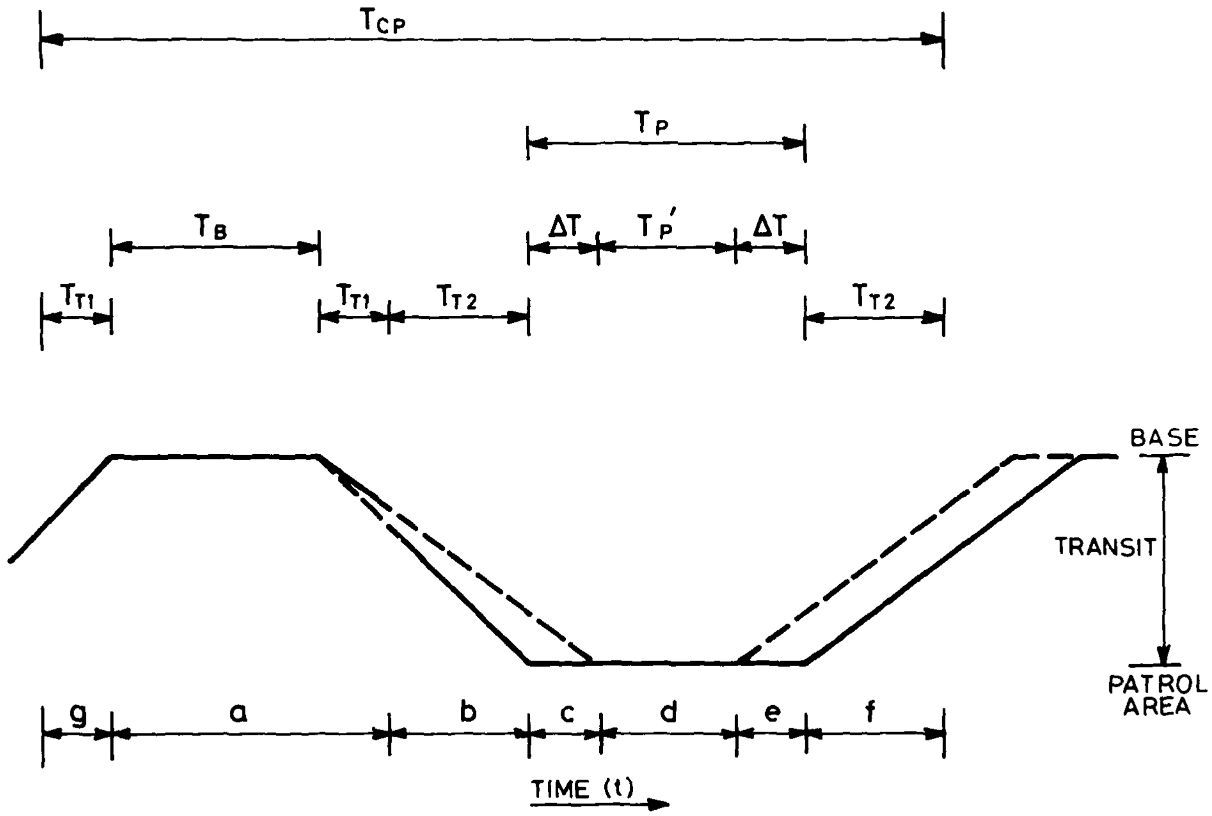

Fig. C·1. Partitioning of submarine operational cycle used in model derivation.

$$\text{where} \quad Z(N)_b = 0 \qquad \text{if } N \neq 0 \tag{C.3}$$

$$= \frac{1}{T_{T2}} \cdot G(T_{T2}, T_{T2}, \Delta T) - \Delta T \qquad \text{if } N = 0$$

The function $Z(N)$ is a correction that must be applied when $T_L < T_{CP}$ (i.e. $N = 0$)

c. Here, the minefield is laid when the submarine is in the first ΔT days of its time on station. The first occasion on station is therefore of length between $T_P' + \Delta T$ and T_P' days with an expectation value of $T_P' + \frac{\Delta T}{2}$ days. The solution is

$$E(n)_c = \frac{(NT_P' + \Delta T/2)}{T_P} + \frac{1}{T_P} \cdot \left[\frac{1}{\Delta T} \cdot G(\Delta T, \Delta T, T_P') + Z(N)_c\right] \tag{C.4}$$

$$\text{where} \quad Z(N)_c = 0 \qquad \text{if } N \neq 0 \tag{C.5}$$

$$= \frac{1}{\Delta T} \cdot G(0, \Delta T, \Delta T) - \Delta T \qquad \text{if } N = 0$$

d. Here, the minefield is laid when the submarine is already on station (see Fig. C.1). The solution (using eqn. F.2) is

$$E(n)_d = N\frac{T_P'}{T_P} + \frac{1}{T_P} \cdot \left[\frac{1}{T_P'} \cdot G(0, T_P', T_P') - \frac{T_P'}{2} + \frac{1}{T_P'} \cdot G(T_{CP}, T_P', T_P')\right] . \tag{C.6}$$

e. Here, the submarine has already completed $T_P' + \Delta T$ days on station with the intention of staying T_P days on station. As soon as minelaying begins, the submarine returns to its base and is forced to take the detour. (In practice, the submarine is unlikely to be immediately aware that mining has begun.) Note that in this case, the submarine exceeds its maximum endurance, T_{PL}, by up to ΔT days; in order to prevent the operational cycle length, T_{CP}, being exceeded, it is assumed that the time the submarine spends at its base, on its return,

is correspondingly shortened to

$$T_B' = T_B - x$$

where $0 \leq x \leq \Delta T$, and the condition $\Delta T \leq T_B$ must be satisfied. The penalty to the submarine as a result of the shortened time spent at its base (this occurs only once) is difficult to assess and is not included in this model.
The solution is

$$E(n)_e = N\frac{T_P'}{T_P} + \frac{G(T_B + 2T_T + 2\Delta T,\ \Delta T,\ T_P')}{T_P.\Delta T} \tag{C.7}$$

f. Here, the submarine has not passed the minefield area before mining begins, the detour must therefore be taken. The submarine endurance is exceeded by ΔT days, and the time in base is shortened to

$$T_B' = T_B - \Delta T \quad .$$

The solution is

$$E(n)_f = N\frac{T_P'}{T_P} + \frac{G(T_B + 2T_T + \Delta T,\ T_{T2},\ T_P')}{T_P.T_{T2}} \tag{C.8}$$

g. Here, the submarine has already passed the minefield area when mining begins; no detour is taken. The solution is

$$E(n)_g = N\frac{T_P'}{T_P} + \frac{G(T_B + T_T + \Delta T + T_{T1},\ T_{T1},\ T_P')}{T_P.T_{T1}} \tag{C.9}$$

The expectation value, $E(n)$, for a complete submarine operational cycle is obtained by taking the weighted average of the results for parts a. to g. :

$$E(n) = \frac{1}{T_{CP}}\Big[(T_B + T_{T1}) . E(n)_a + T_{T2}. E(n)_b + \Delta T. E(n)_c + T_P' . E(n)_d + \Delta T. E(n)_e + T_{T2} . E(n)_f + T_{T1} . E(n)_g\Big] . \tag{C.10}$$

Since it can be shown that

$$G(T_{CP} - T_P' - T_T - \Delta T,\ T_{CP} - T_T - \Delta T,\ T_P') = G(T_B + T_T + \Delta T,\ T_B + T_{T1},\ T_P')$$

$$+ G(T_{T2} + \Delta T,\ T_{T2},\ T_P') + G(\Delta T, \Delta T,\ T_P') + G(0,\ T_P',\ T_P') \qquad (C.11)$$

and that

$$G(T_{CP},\ T_P' + T_T + \Delta T,\ T_P') = G(T_{CP}, T_P',\ T_P') + G(T_B + 2T_T + 2\Delta T, \Delta T,\ T_P')$$

$$+ G(T_B + 2T_T + \Delta T,\ T_{T2},\ T_P') + G(T_B + T_T + \Delta T + T_{T1},\ T_{T1},\ T_P') \qquad (C.12)$$

the solution to eqn. C.10 simplies to

$$E(n) = N\frac{T_P'}{T_P} + \frac{\Delta T}{T_P . T_{CP}}\left[T_{T2} + \frac{\Delta T}{2}\right] + \frac{1}{T_P . T_{CP}}\Big\{G(T_{CP} - T_P' - T_T - \Delta T,\ T_{CP} - T_T - \Delta T, T_P')$$

$$+ G(T_{CP},\ T_P' + T_T + \Delta T,\ T_P') - \frac{T_P'^2}{2} - Z(N)\Big\} \qquad (C.13)$$

where $Z(N)$ is given by

$$Z(N) = T_{T2} . Z(N)_b + \Delta T . Z(N)_c$$

i.e. $\quad Z(N) = 0 \qquad\qquad$ if $N \neq 0 \qquad$ (C.14)

$$= \Delta T(T_{T2} + \Delta T) - G(T_{T2},\ T_{T2} + \Delta T,\ \Delta T) \quad \text{if} \quad N = 0$$

This equation can be rewritten as

$$Z(N) = \Delta T(T_{T2} + \frac{\Delta T}{2} - T_L) \qquad T_L \leq T_{T2} \qquad (C.15)$$

$$= \frac{1}{2}(T_{T2} + \Delta T - T_L)^2 \qquad T_{T2} < T_L < T_{T2} + \Delta T$$

$$= 0 \qquad T_L \geq T_{T2} + \Delta T$$

It can be further shown that

$$G(T_{CP} - T_P' - T_T - \Delta T, T_{CP} - T_T - \Delta T, T_P') + G(T_{CP}, T_P' + T_T + \Delta T, T_P') = \frac{T_P'^2}{2} + T_P' T_R \qquad (C.16)$$

The use of eqns. C.15 and C.16 to simplify eqn. C.13, gives

$$E(n) = \frac{T_L}{T_{CP} \cdot T_P} \cdot (T_P' + \Delta T) \qquad T_L \leq T_{T2} \quad (C.17)$$

$$= \frac{T_L}{T_{CP} \cdot T_P} \cdot (T_P' + T_{T2} + \Delta T - \frac{T_L}{2}) + \frac{T_{T2}^2}{2.T_{CP} \cdot T_P} \qquad T_{T2} < T_L < T_{T2} + \Delta T$$

$$= \frac{T_L \cdot T_P'}{T_{CP} \cdot T_P} + \frac{\Delta T}{T_{CP} \cdot T_P} (T_{T2} + \frac{\Delta T}{2}) \qquad T_L \geq T_{T2} + \Delta T$$

Equation 3.36 is obtained by substituting eqns. C.17 and 3.3 into eqn. 3.4.

C.2 LOWER AND UPPER BOUNDS

Lower and upper bounds, n_ℓ and n_u, for the number of times a submarine can spend on station during the mining campaign, are determined only for the case where $T_L = NT_{CP}$. Since $T_R = 0$, the second term in each of eqns. C.1, C.2, C.4, C.6, C.7, C.9, and C.10 is zero. Examination of these equations shows that $E(n)$ is a maximum in part b. (Section C.1) and a minimum in parts a., and d. to g. Thus

$$n_\ell = N \frac{T_P'}{T_P} \qquad (C.18)$$

and

$$n_u = \frac{1}{T_P} (NT_P' + \Delta T) \qquad (C.19)$$

Results for ΔS_ℓ and ΔS_u (eqns. 3.37 and 3.38) are obtained by substituting eqns. C.19 and C.18 into eqns. 3.5 and 3.6, respectively, with $T_L = NT_{CP}$.

ANNEX D

MINEFIELD IS CLEARED - DERIVATION OF MODEL

The concept of this model is described in Subsection 3.4. The solution for the expected value, $\overline{\Delta S}$, (eqn. 3.44) is derived in Section D.1, while the lower and upper bounds, ΔS_ℓ and ΔS_u , (eqns 3.47 and 3.48) are derived in Section D.2. A simpler version of eqn. 3.44, but which is only valid for certain values of T_W , is derived in Section D.3. The expected waiting time, T_W , and standard deviation (eqns. 3.40 and 3.41) are derived in Section D.4.

D.1 EXPECTED VALUE

The solution for the expected value, $\overline{\Delta S}$, is derived here for the general case where T_L can assume any value. The submarine operational cycle (Annex A) is divided into a number of parts, depending on the submarine position at the start of the mining campaign; this is shown in Fig. D.1. It can be seen that submarines in region a. will suffer a delay $2T_W$ before reaching the patrol area, those in region b. will suffer a delay T_W , and those in region c. will suffer no delay. The expected number of patrols during time T_L is given by

$$E(n) = \frac{1}{T_{CP}} \left[\int_a n(t)\, dt + \int_b n(t)\, dt + \int_c n(t)\, dt \right] \quad \text{(D.1)}$$

where the three integrals are over regions a., b., and c. respectively, and n(t) is the number of patrols if the minefield is laid at time t and the length of the mining campaign is T_L . Note that no integration takes place over regions d. and e. since it is impossible for the submarine to initially be in those regions, eqn. D.1 is more easily solved by solving

$$E(n) = \frac{1}{T_{CP}} \left[\int_0^{T_{CP}'} n(t)\, dt - \int_d n(t)\, dt - \int_e n(t)\, dt \right] \quad \text{(D.2)}$$

where T_{CP}' is defined in eqn. 3.39.

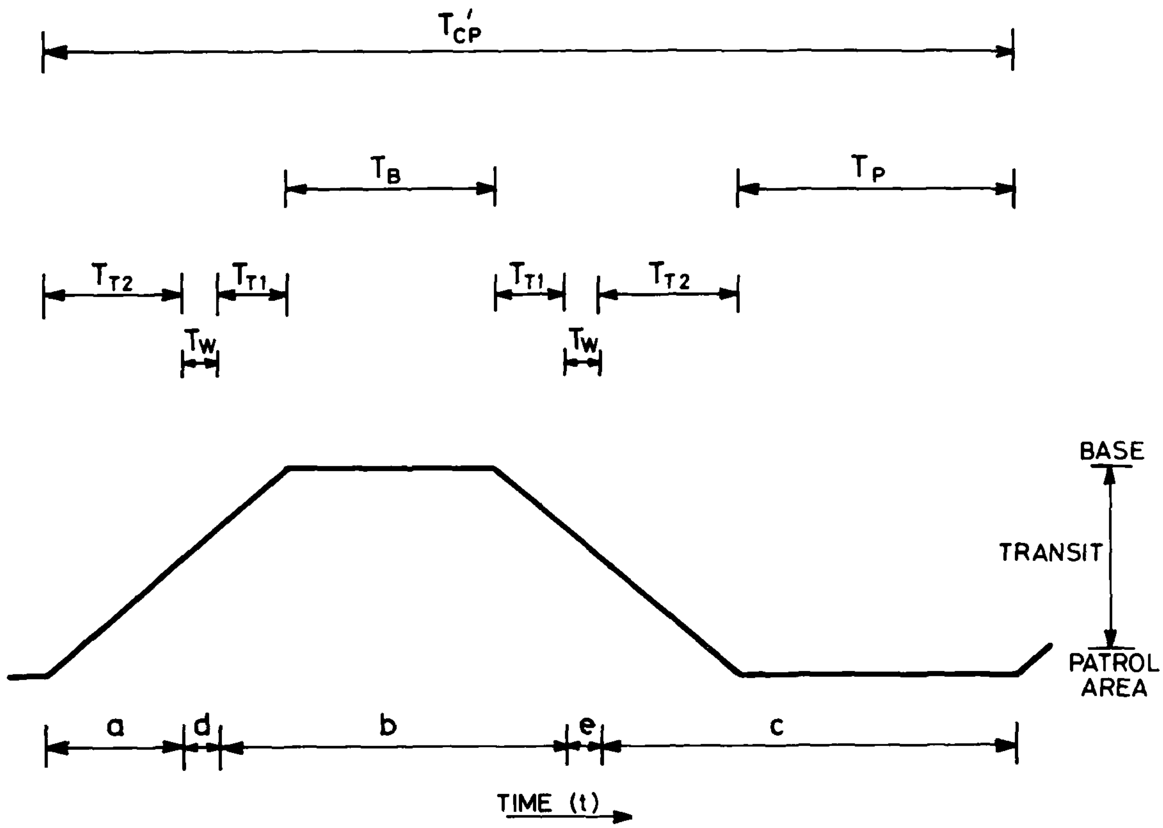

Fig. D-1. Partitioning of submarine operational cycle used in model derivation.

The solution for each of these integrals is derived below:

a. The solution to the first integral in eqn. D.2 can be expressed in terms of the function G(y,z,m) described in Annex F (note also the use of eqn. F.2),

$$\int_0^{T_{CP}'} n(t)\ dt = N'T_{CP}' + \frac{1}{T_P}\left[G(T_{CP}' - T_P,\ T_{CP}',\ T_P) - \frac{T_P^{\ 2}}{2} + G(T_{CP}',\ T_P, T_P) \right] \quad \text{(D.3)}$$

where N' and T_R' are defined in eqns. 3.43 and 3.44. The second term in this equation can be shown to be equal to T_R'. Thus eqn. D.3 becomes

$$\int_0^{T_{CP}'} n(t)\ dt \ = \ N'T_{CP}' + T_R' \ = \ T_L \quad . \qquad \text{(D.4)}$$

b. The solution to the second integral in eqn. D.2 is given by

$$\int_d n(t)\ dt = N'T_W + \frac{1}{T_P} \cdot G(T_{CP}' - T_P - T_{T2},\ T_W,\ T_P) \quad . \qquad \text{(D.5)}$$

c. The solution to the third integral in eqn. D.2 is given by

$$\int_e n(t)\ dt = N'T_W + \frac{1}{T_P} \cdot G(T_{T2} + T_W,\ T_W,\ T_P) \quad . \qquad \text{(D.6)}$$

Substitution of eqns. D.4, D.5 and D.6 into eqn. D.2 gives the solution for E(n),

$$E(n) \ = \ \frac{T_L}{T_{CP}} - \frac{2N'T_W}{T_{CP}} - \frac{1}{T_{CP} \cdot T_P}\Big[G(T_{CP}' - T_P - T_{T2},\ T_W,\ T_P)$$

$$+ G(T_{T2} + T_W,\ T_W,\ T_P) \Big] \quad . \qquad \text{(D.7)}$$

The third term in eqn. D.7 is zero when $T_R' = 0$.

Equation 3.44 is obtained by substituting eqns. D.7 and 3.3 into 3.4.

D.2 LOWER AND UPPER BOUNDS

Lower and upper bounds, n_ℓ and n_u, for the number of times a submarine can spend on station during the mining campaign, are determined only for the case where $T_L = NT_{CP}$. Examination of Fig. D.1 shows that the number of times on station is minimised when the mining campaign begins just as the submarine completes its time on station and begins the return transit to its base. In this case,

$$n_\ell = N' + \frac{1}{T_P} \cdot \text{Max}\left[0;\ T_R' + T_P - T_{CP}'\right] \quad . \qquad (D.8)$$

The number of times on station is maximised when mining begins just as the submarine completes its outward transit and begins its time on station. In this case,

$$n_u = N' + \frac{1}{T_P} \cdot \text{Min}\left[T_p\ ;\ T_R'\right] \quad . \qquad (D.9)$$

Results for ΔS_ℓ and ΔS_u (eqns. 3.45 and 3.46) are obtained by substituting eqns. D.9 and D.8 into eqns. 3.5 and 3.6, repectively, with $T_L = NT_{CP}$.

D.3 SIMPLIFICATION OF $\overline{\Delta S}$

The derivation of eqn. 3.50 and condition 3.51 is presented in this section. Note that this derivation applies only when $T_L = NT_{CP}$.

Examination of eqn. 3.44 shows that it is identical t[illegible]n. 3.50 if

a. $N' = N - n$ and $G_1 = G_2 = T_W T_P$

or if

b. $N' = N - (n - 1)$ and $G_1 = G_2 = 0$

where G_1 and G_2 represent the two functions $G(y,z,m)$ in the second

term of eqn. 3.44. The derivation consists of determining the conditions imposed by cases a. and b. .

Case a. :

An initial restriction on the values of T_W is obtained from eqn. 3.45,

$$N - n \leq \frac{NT_{CP}}{T_{CP} + 2T_W} < N - n + 1 \qquad \text{(D.10)}$$

and rearranging,

$$\frac{(n-1)\,T_{CP}}{2\left[N - (n-1)\right]} < T_W \leq \frac{nT_{CP}}{2\left[N - n\right]} \quad . \qquad \text{(D.11)}$$

For $G_1 = T_W T_P$ it is necessary that

$$T_{CP} - T_P - T_{T2} + 2T_W \leq T_R' - T_P \qquad \text{(D.12)}$$

(see eqn. F.3) ; rearranging (using eqn. 3.46) gives

$$T_W \leq \frac{(n-1)\,T_{CP} + T_{T2}}{2\left[N - (n-1)\right]} \qquad \text{(D.13)}$$

as the required condition. Similarly for $G_2 = T_W T_P$ it is necessary that

$$T_{T2} + T_W \leq T_R' - T_P \qquad \text{(D.14)}$$

so that

$$T_W \leq \frac{nT_{CP} - T_P - T_{T2}}{2\left[N - (n-1)\right] - 1} \qquad \text{(D.15)}$$

is the necessary condition.

It can be shown that conditions D.11, D.13 and D.15 are simultaneously satisfied if

$$\frac{(n-1)\,T_{CP}}{2\left[N - (n-1)\right]} < T_W \leq \frac{(n-1)\,T_{CP} + T_{T2}}{2\left[N - (n-1)\right]} \quad . \qquad \text{(D.16)}$$

Case b. :

A restriction on the values of T_W is obtained from eqn. 3.45,

$$N - (n - 1) \leq \frac{NT_{CP}}{T_{CP} + 2T_W} < N - (n - 1) + 1 \qquad (D.17)$$

which can be rearranged to give

$$\frac{(n - 1)\ T_{CP} - T_{CP}}{2\left[N - (n - 1)\right] + 2} < T_W \leq \frac{(n - 1)\ T_{CP}}{2\left[N - (n - 1)\right]} \qquad (D.18)$$

For $G_1 = 0$ it is necessary that

$$T_{CP} - T_P - T_{T2} + 2T_W \geq T_R' + T_W \qquad (D.19)$$

i.e.

$$T_W \geq \frac{(n - 2)T_{CP} + T_P + T_{T2}}{2\left[N - (n - 1)\right] + 1} \qquad (D.20)$$

Similarly, for $G_2 = 0$ it is necessary that

$$T_{T2} + T_W \geq T_R' + T_W \qquad (D.21)$$

i.e.

$$T_W \geq \frac{(n - 1)\ T_{CP} - T_{T2}}{2\left[N - (n - 1)\right]} \qquad (D.22)$$

It can be shown that conditions D.18, D.20 and D.22 are simultaneously satisfied if

$$\frac{(n - 1)\ T_{CP} - T_{T2}}{2\left[N - (n - 1)\right]} \leq T_W \leq \frac{(n - 1)\ T_{CP}}{2\left[N - (n - 1)\right]} \qquad (D.23)$$

which is the condition imposed by case b. .

It is a simple matter to combine conditions D.16 and D.23 for cases a. and b. respectively into a single condition

$$\frac{(n-1)\,T_{CP} - T_{T2}}{2\left[N - (n-1)\right]} \leq T_W \leq \frac{(n-1)\,T_{CP} + T_{T2}}{2\left[N - (n-1)\right]} \quad . \tag{D.24}$$

This is the condition on T_W that must be satisfied for eqn. 3.50 to be valid.

D.4 EXPECTED WAITING TIME

The expected time that submarines must wait before the minefield is cleared depends on the time, T_C, taken to clear the minefield, the time, T_{RP}, between consecutive minefield replenishments by the minelayer, and on the time, t, when the submarine arrives at the minefield. Note that T_C must be less than T_{RP}, otherwise a channel is never completely cleared.

If a particular submarine arrives at the minefield at time t, then it must wait for a time,

$$\begin{aligned} t_W &= T_C - t && \text{for} \quad t < T_C \\ &= 0 && \text{for} \quad T_C \leq t < T_{RP} \end{aligned} \tag{D.25}$$

before it can cross the minefield. This situation is depicted in Fig. D.2. Clearly, if the submarine arrives at time t_a, then the minefield is already partly cleared and it will only have to wait $T_C - t_a$ days. If the submarine arrives at time t_b, then the minefield is already clear and there is no waiting time.

If the submarine arrival time is random (i.e. the submarine is unable to plan its arrival to coincide with the time when the minefield is clear) and thus uniformly distributed $(0 \leq t \leq T_{RP})$, then the expected waiting time is

$$E(t_W) = \frac{1}{T_{RP}} \cdot \int_0^{T_{RP}} t_W \, dt \tag{D.26}$$

where t_W is given by eqn. D.25. The solution of eqn. D.26 is

$$E(t_W) = T_W = \frac{T_C^{\,2}}{2T_{RP}} \tag{D.27}$$

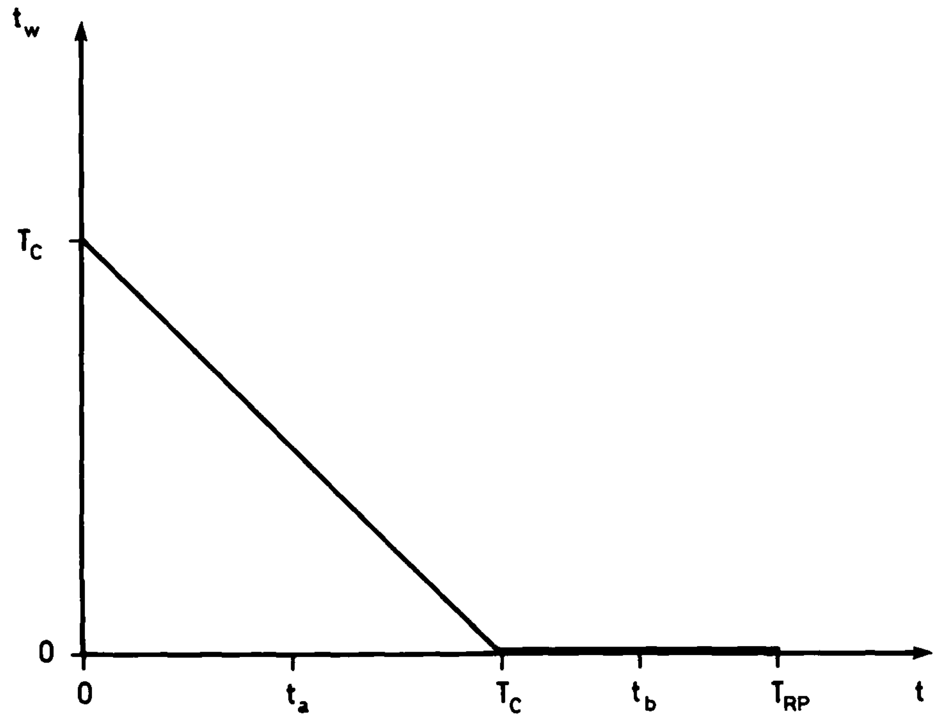

Fig. D·2. Submarine waiting time as a function of arrival time.

The variance of the waiting time is given by

$$\mathrm{Var}(t_W) = E\,(t_W{}^2) - \left[E(t_W) \right]^2$$

$$= \frac{1}{T_{RP}} \int_0^{T_{RP}} t_W{}^2 \, dt - \frac{T_C{}^4}{4T_{RP}{}^2}$$

$$= \frac{T_C{}^4}{4T_{RP}{}^2} \left[\frac{4T_{RP}}{3T_C} - 1 \right] \qquad (D.29)$$

The standard deviation is therefore,

$$\sigma_W = \frac{T_C{}^2}{2T_{RP}} \left[\frac{4T_{RP}}{3\ T_C} - 1 \right]^{\frac{1}{2}} \qquad (D.30)$$

or

$$\sigma_W = T_W \left[\frac{4T_{RP}}{3\ T_C} - 1 \right]^{\frac{1}{2}} \qquad (D.31)$$

ANNEX E

COVERT MINING - DERIVATION OF FORMULAE

The derivation of the expected time, T_D, for the first submarine casualty due to a covert minefield, and its standard deviation, σ_D, is based on the geometric distribution (eqn. 3.7). The probability of a submarine casualty on the i^{th} crossing is

$$\Pr(i) = p\,q^{i-1} \tag{E.1}$$

where $q = 1 - p$, and p is the probability of a casualty per crossing. The number of the crossing on which the first casualty is expected to occur is given by

$$E(i) = \sum_{i=1}^{\infty} i\,p\,q^{i-1} \tag{E.2}$$

with the solution (see Ref. 3, and series No. 5 Ref. 2)

$$E(i) = \frac{1}{p} \quad . \tag{E.3}$$

The variance is given by

$$\mathrm{Var}(i) = E(i^2) - \left[E(i)\right]^2 \tag{E.4}$$

i.e.

$$\mathrm{Var}(i) = \sum_{i=1}^{\infty} i^2\,p\,q^{i-1} - \left[\sum_{i=1}^{\infty} i\,p\,q^{i-1}\right]^2 \tag{E.5}$$

This expression can be simplified (using series No. 1113 in Ref. 2) to

$$\mathrm{Var}(i) = \frac{q}{p^2} \tag{E.6}$$

If S submarine regularly ᵔss the minefield there is a minefield crossing every

$$T_{CR} = \frac{T_{CP}}{2\,S} \quad \text{days.} \tag{E.7}$$

Therefore, T_D is given by

$$T_D = \frac{T_{CP}}{2\,p\,S}$$

with standard deviation

$$\sigma_D = \frac{T_{CP}}{2\,p\,S} \cdot \sqrt{q} \qquad (E.8)$$

or

$$\sigma_D = T_D \cdot \sqrt{q} \quad . \qquad (E.9)$$

ANNEX F

DERIVATION OF FUNCTION G(y,z,m)

F.1 INTRODUCTION

The purpose of function G(y,z,m) is to account for the contribution of T_R when evaluating the total expected number of days on patrol, E(n). T_R is the number of days remaining at the end of the mining campaign after all complete submarine operational cycles have been taken into account (see eqn. 3.1 for definition). The function G(y,z,m) is defined by

$$G(y,z,m) = \int_0^z f(x)\,dx \tag{F.1}$$

where

$$f(x) = \begin{cases} 0 & \text{for } x \leq y - T_R \\ x + T_R - y & y - T_R < x < m + y - T_R \\ m & x \geq m + y - T_R \end{cases}$$

Note that $0 \leq x \leq z$, and $y \geq 0$.

It can be seen from Fig. F.1 that f(x) is the number of patrol days encompassed in the range $t = 0$ to $t = T_R + x$. Thus $G(y,z,m)/z$ is the expected number of patrol days encompassed in the range $t = 0$ to $t = T_R + z$. The expected number of patrol days due to T_R only is given by

$$E'(n, T_R) = \frac{1}{z}\left[G(y,z,m) - \frac{1}{2}\ K(z)^2 \right] \tag{F.2}$$

where

$$K(z) = \begin{cases} 0 & \text{for } z \leq y \\ z - y & y < z < m + y \\ m & z \geq m + y \end{cases}$$

The function K(z) accounts for situations where $x > y$. (Note that eqn.

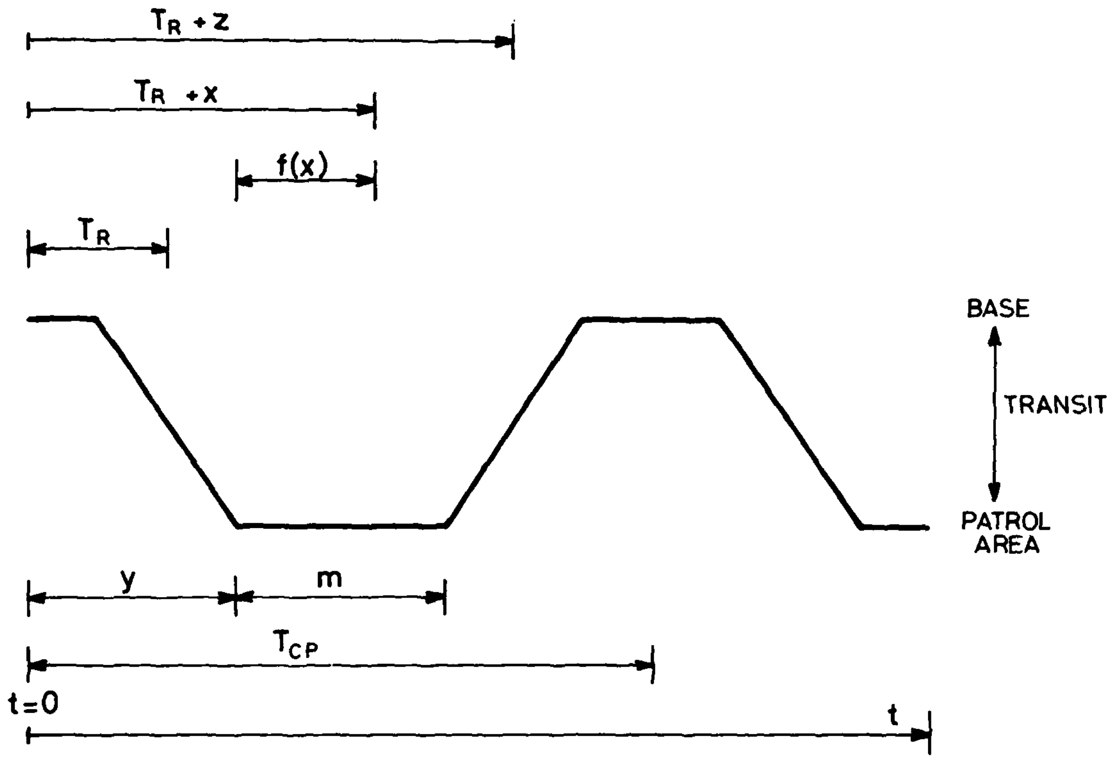

Fig. F-1. Submarine patrol cycle.

F.2 is not used explicity in model derivation; it is presented here to illustrate the meaning of function $G(y,z,m)$.)

Function $G(y,z,m)$ is given by,

$$
\begin{aligned}
G(y,z,m) &= mz && \text{for } y \leq T_R - m && \text{(F.3)}\\
&= mz - \tfrac{1}{2}\left[T_R - y - m\right]^2 && T_R - m < y < T_R + z - m - J\\
&= (z - J)\left[T_R - y + \tfrac{1}{2}z + \tfrac{1}{2}J\right] && T_R + z - m - J \leq y \leq T_R + J\\
&= \tfrac{1}{2}\left[T_R - y + z\right]^2 && T_R + J < y < T_R + z\\
&= 0 && y \geq T_R + z
\end{aligned}
$$

where $J = \text{Max}\left[0;\ z - m\right]$

and $y \geq 0$

Note that $G(y,z,m) = 0$ when $z = 0$.

F.2 DERIVATION

$G(y,z,m)$ is derived by application of eqn. F.1 (see also Fig. F.1). It is convenient to consider two cases:

1. <u>$z \leq m \quad (J = 0)$</u>

a. if $T_R + z \leq y$, i.e. $y \geq T_R + z$

then $G = \int_0^z 0\,dx = 0$

b. if $y < T_R + z < y + z$, i.e. $T_R \leq y \leq T_R + z$

then $G = \int_0^{y-T_R} 0\,dx + \int_{y-T_R}^{z} x + T_R - y\,dx = \frac{1}{2}\left[T_R - y + z\right]^2$

c. if $y + z \leq T_R + z \leq y + m$, i.e. $T_R + z - m \leq y \leq T_R$

then $G = \int_0^z x + T_R - y\,dx = z\left[T_R - y + \frac{1}{2}z\right]$

d. if $y + m < T_R + z < y + z + m$, i.e. $T_R - m < y < T_R + z - m$

$$\text{then } G = \int_0^{y+m-T_R} x + T_R - y \; dx + \int_{y+m-T_R}^{z} m \; dx$$

$$= mz - \frac{1}{2}\left[T_R - y - m \right]$$

e. if $T_R \geq y + m$, i.e. $y \leq T_R - m$

$$\text{then } G = \int_0^{z} m \; dx = mz$$

2. <u>$z > m$ $(J = z - m)$</u>

a. same as for a. above.

b. if $y < T_R + z < y + m$, i.e. $T_R + z - m < y < T_R + z$

then $G = \frac{1}{2}\left[T_R - y + z \right]^2$ as before

c. if $y + m \leq T_R + z \leq y + z$, i.e. $T_R \leq y \leq T_R + z - m$

$$\text{then } G = \int_{y-T_R}^{y+m-T_R} x + T_R - y \; dx + \int_{y+m-T_R}^{z} m \; dx$$

$$= m\left[T_R - y + z - \frac{1}{2} m \right]$$

d. if $y + z < T_R + z < y + z + m$, i.e. $T_R - m < y < T_R$

then $G = mz - \frac{1}{2}\left[T_R - y - m \right]^2$ as before

e. same as for e. above

By introducing $J = \text{Max}\left[0; z - m \right]$, cases 1. and 2. can be combined to give eqn. F.3.

NOTATION

ASW	Anti-submarine warfare.
f	Fraction of the time spent on station, that remains to be completed.
i	Minefield crossing number.
MCM	Mine countermeasures.
n	Number of times a submarine is on station in the patrol area during a mining campaign of length T_L .
n'	Number of times a submarine is on station in the patrol area during a time T_L , if no mining takes place.
n''	Number of times a submarine is on station in the patrol area during time T_L , if it decides to wait for minefield clearance.
n_ℓ	Lower bound for n.
n_u	Upper bound for n.
N	Number of complete submarine operational cycles during the mining campaign.
N'	Number of complete submarine operational cycles during the mining campaign if submarines decide to wait for minefield clearance.
p	Probability of a submarine casualty per minefield crossing.
p_M	Probability of a submarine casualty per minefield crossing, assuming a constant minefield.
$Pr(i)$	Probability of a submarine casualty on its i^{th} minefield crossing.
P(n patrols)	Probability of a submarine completing exactly n patrols.
q	Probability of a submarine surviving a minefield crossing.
σ_D	Standard deviation for T_D .
σ_W	Standard deviation for T_W .
S	Number of submarines.
S_o	Initial number of submarines.
$\overline{\Delta S}$	Expectation value for the average reduction in submarines on station.
ΔS_ℓ	Lower bound for the reduction in submarines on station.
ΔS_u	Upper bound for the reduction in submarines on station
t	Time.
t_W	Submarine waiting time for minefield clearance.

T_B	Length of time spent by a submarine at its base.
T_C	Time taken to clear a channel through the minefield.
T_{CP}	Submarine operational cycle length.
T_{CR}	Time between successive minefield crossings by submarines.
T_D	Expected time for the first submarine casualty in a covert minefield.
T_L	Length of the mining campaign.
T_P	Length of time spent by a submarine on station in the patrol area.
T_{PL}	Submarine patrol length (submarine endurance).
T_R	Number of days remaining at the end of the mining campaign after all (N) complete submarine operational cycles have been taken into account.
T_R'	Number of days remaining at the end of the mining campaign after all (N') complete submarine operational cycles have been taken into account. The dashed superscript indicates that a submarine operational cycle length of $T_{CP} + 2T_W$ is used.
T_{RP}	Time between consecutive minefield replenishments.
T_T	One-way submarine transit time between the submarine base and the patrol area.
T_{T1}	One-way submarine transit time between the submarine base and the minefield.
T_{T2}	One-way submarine transit time between the minefield and the patrol area.
T_W	Expected submarine waiting time during minefield clearance.